Simplifying and extending the $AdS_5 \times S^5$ pure spinor formalism

JHEP09(2009)051

(http://iopscience.iop.org/1126-6708/2009/09/051)

Published by IOP Publishing for SISSA

Received: *February 15, 2009*
Revised: *July 16, 2009*
Accepted: *August 27, 2009*
Published: *September 9, 2009*

Simplifying and extending the $AdS_5 \times S^5$ pure spinor formalism

Nathan Berkovits

Instituto de Física Teórica, São Paulo State University,
Rua Pamplona 145, 01405-900, São Paulo, SP, Brasil

E-mail: nberkovi@ift.unesp.br

Abstract: Although the $AdS_5 \times S^5$ worldsheet action is not quadratic, some features of the pure spinor formalism are simpler in an $AdS_5 \times S^5$ background than in a flat background. The BRST operator acts geometrically, the left and right-moving pure spinor ghosts can be treated as complex conjugates, the zero mode measure factor is trivial, and the b ghost does not require non-minimal fields.

Furthermore, a topological version of the $AdS_5 \times S^5$ action with the same worldsheet variables and BRST operator can be constructed by gauge-fixing a G/G principal chiral model where $G = \mathrm{PSU}(2,2|4)$. This topological model is argued to describe the zero radius limit that is dual to free $\mathcal{N} = 4$ super-Yang-Mills and can also be interpreted as an "unbroken phase" of superstring theory.

Keywords: AdS-CFT Correspondence, Topological Strings

ArXiv ePrint: 0812.5074

doi:10.1088/1126-6708/2009/09/051

JHEP09(2009)051

Contents

JHEP09(2009)051

1 Introduction

Up to now, the only superstring formalism suitable for covariantly quantizing the $AdS_5 \times S^5$ background is the pure spinor formalism [1]. Because of the Ramond-Ramond flux, the Ramond-Neveu-Schwarz formalism cannot describe this background. Although the covariant Green-Schwarz formalism can classically describe the $AdS_5 \times S^5$ background, this formalism has only been quantized in light-cone gauge by expanding around classical solutions which break the target-space $\mathrm{PSU}(2,2|4)$ invariance. It should be noted that for computing the physical spectrum, the light-cone Green-Schwarz formalism is probably the most convenient since it includes only physical degrees of freedom and does not require ghosts. However, for computing scattering amplitudes or for describing the spectrum in a $\mathrm{PSU}(2,2|4)$-invariant manner, the pure spinor formalism is expected to be more convenient since it manifestly preserves all symmetries.

In a flat target-space background, the worldsheet action in the pure spinor formalism is quadratic and it is easy to compute scattering amplitudes using the free-field OPE's of the worldsheet fields. However, in an $AdS_5 \times S^5$ background, the worldsheet action is [2]

$$S = \int d^2 z \Big[\frac{1}{2}\eta_{ab} J^a \overline{J}^b - \eta_{\alpha\widehat{\beta}} \Big(\frac{3}{4} J^{\widehat{\beta}} \overline{J}^{\alpha} + \frac{1}{4} \overline{J}^{\widehat{\beta}} J^{\alpha} \Big) \\ - w_\alpha \overline{\nabla} \lambda^\alpha + \widehat{w}_{\widehat{\alpha}} \nabla \widehat{\lambda}^{\widehat{\alpha}} - \frac{1}{4} \eta_{[ab][cd]} (w\gamma^{ab}\lambda)(\widehat{w}\gamma^{cd}\widehat{\lambda}) \Big] \tag{1.1}$$

where $J^A = (g^{-1}\partial g)^A$ and $\overline{J}^A = (g^{-1}\overline{\partial} g)^A$ are the Metsaev-Tseytlin left-invariant currents [3], $A = (a, \alpha, \widehat{\alpha}, [ab])$ are the PSU(2,2|4) Lie-algebra indices, g takes values in the $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times \mathrm{SO}(5)}$ coset, $(\lambda^\alpha, w_\alpha)$ and $(\widehat{\lambda}^{\widehat{\alpha}}, \widehat{w}_{\widehat{\alpha}})$ are the left and right-moving pure spinor variables, and $(\eta_{ab}, \eta_{\alpha\widehat{\beta}}, \eta_{[ab][cd]})$ are the nonvanishing components of the PSU(2,2|4) metric. The global PSU(2,2|4) isometries act on g by left multiplication as $\delta g = \Sigma g$, and these global isometries commute with the BRST transformations which act by right multiplication as

$$Qg = g\ (\lambda^\alpha T_\alpha + \widehat{\lambda}^{\widehat{\alpha}} T_{\widehat{\alpha}}) \tag{1.2}$$

where T_α and $T_{\widehat{\alpha}}$ are the fermionic generators of PSU(2,2|4). Since the J^A currents are not holomorphic, it is difficult to compute OPE's and scattering amplitudes in an $AdS_5 \times S^5$ background.

Nevertheless, it will be shown in the first half of this paper that there are several features of the pure spinor formalism in an $AdS_5 \times S^5$ background which are simpler than in a flat background. Unlike the worldsheet Lagrangian in a flat background which transforms by a total derivative under $d = 10$ supersymmetry transformations, the worldsheet Lagrangian of (1.1) is manifestly PSU(2,2|4) invariant. As a consequence, the vertex operator for the zero-momentum dilaton in an $AdS_5 \times S^5$ background is manifestly PSU(2,2|4) invariant and can be expressed as the ghost-number (1,1) operator

$$V^{\mathrm{AdS}} = \eta_{\alpha\widehat{\alpha}} \lambda^\alpha \widehat{\lambda}^{\widehat{\alpha}} \tag{1.3}$$

where $\eta_{\alpha\widehat{\alpha}} \equiv (\gamma^{01234})_{\alpha\widehat{\alpha}}$. On the other hand, the zero-momentum dilaton vertex operator in a flat background is

$$V^{\mathrm{flat}} = (\lambda\gamma^m\theta)(\widehat{\lambda}\gamma_m\widehat{\theta}), \tag{1.4}$$

which transforms under spacetime supersymmetry into a BRST-trivial operator.

Because $(\eta_{\alpha\widehat{\alpha}}\lambda^\alpha\widehat{\lambda}^{\widehat{\alpha}})$ is in the BRST cohomology in an $AdS_5 \times S^5$ background, it is consistent to impose the constraint that $(\eta\lambda\widehat{\lambda})$ is non-vanishing and to extend the Hilbert space to include states which depend on inverse powers of $(\eta\lambda\widehat{\lambda})$. Note that in a flat background, $(\eta\lambda\widehat{\lambda})$ is not in the cohomology and can be written as $(\eta\lambda\widehat{\lambda}) = Q(\eta_{\alpha\widehat{\alpha}}\theta^\alpha\widehat{\lambda}^{\widehat{\alpha}})$. So in a flat background, such an extension of the Hilbert space would trivialize the cohomology because of the state $W = (\eta\lambda\widehat{\lambda})^{-1}\eta_{\beta\widehat{\beta}}\theta^\beta\widehat{\lambda}^{\widehat{\beta}}$ satisfying $QW = 1$, which would imply that any BRST-closed state V could be written as $V = Q(WV)$.

After extending the Hilbert space in this manner and interpreting λ^α and $\eta_{\alpha\widehat{\alpha}}\widehat{\lambda}^{\widehat{\alpha}}$ as complex conjugates, it is straightforward to define functional integration over the pure spinor variables. Unlike in a flat background where one needs to introduce additional "non-minimal" variables to functionally integrate over pure spinors [4] [5], there is no need to introduce non-minimal variables in an $AdS_5 \times S^5$ background. In some sense, the non-holomorphic structure of the $AdS_5 \times S^5$ sigma model automatically regularizes the 0/0 divergences which were regularized in a flat background by the non-minimal variables.

Since there are no non-minimal variables, the zero mode measure factor and the composite b ghost are simpler in an $AdS_5 \times S^5$ background than in a flat background. In a flat background, the tree-level zero mode measure factor is

$$\langle f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda})\rangle = \int d^{10}x \int (d^5\theta)_{\alpha_1\ldots\alpha_5}(d^5\widehat{\theta})_{\widehat{\alpha}_1\ldots\widehat{\alpha}_5} \tag{1.5}$$
$$\left(\gamma^m\frac{\partial}{\partial\lambda}\right)^{\alpha_1}\left(\gamma^n\frac{\partial}{\partial\lambda}\right)^{\alpha_2}\left(\gamma^p\frac{\partial}{\partial\lambda}\right)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5}\left(\gamma^q\frac{\partial}{\partial\widehat{\lambda}}\right)^{\widehat{\alpha}_1}\left(\gamma^r\frac{\partial}{\partial\widehat{\lambda}}\right)^{\widehat{\alpha}_2}\left(\gamma^s\frac{\partial}{\partial\widehat{\lambda}}\right)^{\widehat{\alpha}_3}(\gamma_{qrs})^{\widehat{\alpha}_4\widehat{\alpha}_5}$$
$$f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda})|_{\theta=\widehat{\theta}=0}$$

and the b ghost satisfying $\{Q,b\} = T$ depends in a complicated manner on the non-minimal variables. In an $AdS_5 \times S^5$ background, the tree-level zero mode measure factor is simply

$$\langle f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda})\rangle = \int d^{10}x \int d^{16}\theta d^{16}\widehat{\theta}\ sdet(E_M^A) \int d\lambda d\widehat{\lambda}\ f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda}) \tag{1.6}$$

where E_M^A is the target-space supervierbein and $\int d\lambda d\widehat{\lambda}$ is a compact integration over the projective pure spinors. And the composite b ghost is

$$b = (\eta\lambda\widehat{\lambda})^{-1}\ \widehat{\lambda}^{\widehat{\alpha}}\left[\frac{1}{2}(\gamma_a J)_{\widehat{\alpha}}J^a + \frac{1}{4}\eta_{\alpha\widehat{\alpha}}N^{ab}(\gamma_{ab}J)^\alpha + \frac{1}{4}\eta_{\alpha\widehat{\alpha}}J_{gh}J^\alpha\right] \tag{1.7}$$

where $(J^\alpha, J^a, J^{\widehat{\alpha}})$ are the left-invariant currents constructed from g, and N^{ab} and J_{gh} are the Lorentz and ghost-currents for λ^α.

It is instructive to consider the pure spinor formalism for the Ramond-Ramond plane-wave background [6] where a partial simplification also occurs. In this background, the operator of (1.3) is replaced with $(\lambda\gamma_{+1234}\widehat{\lambda})$ which only involves the $(\gamma_+\lambda)$ and $(\gamma_+\widehat{\lambda})$ components of the pure spinors. So one still needs to introduce non-minimal variables for the $(\gamma_-\lambda)$ and $(\gamma_-\widehat{\lambda})$ components in order to perform functional integration. This implies that the tree-level measure factor in the plane-wave background involves integration over 18 θ's, as opposed to the 10 θ's in a flat background or the 32 θ's in an $AdS_5 \times S^5$ background.

In principle, these results could be used to compute $AdS_5 \times S^5$ scattering amplitudes without the regularization complications that plague amplitude computations in a flat background [4, 5]. Unfortunately, the difficulties with evaluating OPE's and with constructing explicit vertex operators in an $AdS_5 \times S^5$ background will probably make it hard to compute non-trivial scattering amplitudes at finite AdS radius. Nevertheless, it might eventually be possible to compute amplitudes at infinitesimally small AdS radius and test the Maldacena conjecture in the perturbative super-Yang-Mills regime.

In order to compute superstring amplitudes in this perturbative super-Yang-Mills regime, the first step would be construct a closed string theory that describes the zero radius limit that is dual to free $\mathcal{N} = 4$ super-Yang-Mills theory [7]. Since super-Yang-Mills is a field theory, it is natural to try to describe this zero radius limit using a topological

JHEP09(2009)051

string theory [8]. One recent topological string proposal [9, 10] was constructed from the fermionic coset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,2)\times\mathrm{SO}(6)}$ which was related by a field redefinition to the pure spinor formalism. This topological string theory was later obtained in [11] by gauge-fixing the G/G principal chiral model with $G=\mathrm{PSU}(2,2|4)$, and similar G/G topological models for the zero radius limit have been proposed by A. Polyakov [12] and H. Verlinde [13].

In the second half of this paper, it will be shown that there is an alternative gauge-fixing of the G/G principal chiral model which produces a topological string theory based on the Metsaev-Tseytlin coset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ instead of the fermionic coset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,2)\times\mathrm{SO}(6)}$. This alternative gauge-fixing is related to an $AdS_5\times S^5$ generalization of the "extended pure spinor" formalism proposed by Aisaka and Kazama [14] and, unlike the BRST transformation for the gauge-fixing to the fermionic coset, the BRST transformation using this alternative gauge-fixing is the same as in (1.2).

The worldsheet action of this topological string theory is BRST-trivial and is

$$S_{\mathrm{top}}=\int d^2z\left[\frac{(\lambda\gamma_a\gamma_b\widehat{\lambda})}{2(\eta\lambda\widehat{\lambda})}J^a\overline{J}^b+\eta_{\alpha\widehat{\alpha}}\overline{J}^{\alpha}J^{\widehat{\alpha}}-w_\alpha\overline{\nabla}\lambda^\alpha+\widehat{w}_{\widehat{\alpha}}\nabla\widehat{\lambda}^{\widehat{\alpha}}-\frac{1}{4}\eta_{[ab][cd]}(w\gamma^{ab}\lambda)(\widehat{w}\gamma^{cd}\widehat{\lambda})\right], \tag{1.8}$$

where $J^A=(g^{-1}\partial g)^A$ are the same left-invariant currents constructed from a $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ coset as before. Note that (1.8) differs from the original $AdS_5\times S^5$ action of (1.1) through the $(\lambda^\alpha,\widehat{\lambda}^{\widehat{\alpha}})$ dependence of the first term and the absence of an $\eta_{\alpha\widehat{\alpha}}J^\alpha\overline{J}^{\widehat{\alpha}}$ term.

To show that this topological string theory is the dual to free $\mathcal{N}=4$ super-Yang-Mills, the first step is to show that the BRST cohomology correctly reproduces the single-trace gauge-invariant super-Yang-Mills operators at zero 't Hooft coupling. Since the topological BRST transformations are the same as in the original $AdS_5\times S^5$ model, it is trivial to show that vertex operators for half-BPS states in the original $AdS_5\times S^5$ sigma model are also in the BRST cohomology of the topological sigma model. Vertex operators for non-BPS states can be constructed by acting on half-BPS vertex operators with the σ-dependent transformation

$$\delta g(\sigma)=\Sigma(\sigma)g(\sigma) \tag{1.9}$$

where $\Sigma(\sigma)$ is an arbitrary local $\mathrm{PSU}(2,2|4)$ transformation whose σ-independent modes are the global isometries. These transformations commute with the BRST transformations of (1.2), and when acting on operators of large R-charge, the σ-dependent modes of Σ act like the massive string modes in a plane-wave background by inserting "impurities" in the long operator [15]. Although the σ-dependent transformations of (1.9) do not leave invariant the topological action of (1.8), they only change (1.8) by a BRST-trivial term.

The next step to showing that this topological string theory describes free $\mathcal{N}=4$ super-Yang-Mills is to show that the topological string amplitudes correctly reproduce super-Yang-Mills amplitudes in the limit of small 't Hooft coupling. For string tree amplitudes involving three half-BPS states, these amplitudes are guaranteed to agree since the zero mode measure factor in the topological theory is the same as in (1.6) and since these three-point BPS amplitudes do not depend on the AdS radius.

To show the equivalence of other types of amplitudes, a handwaving argument based on open-closed topological duality will be presented which will hopefully be made more rigor-

JHEP09(2009)051

ous in the future. The argument follows the proposals of [16] and [17, 18] and uses that the open string field theory obtained by putting D_3 branes at the AdS_5 boundary of the topological string reproduces $\mathcal{N}=4$ super-Yang-Mills field theory. Furthermore, it will be argued that perturbing the closed topological action of (1.8) by the vertex operator of (1.1) as

$$S_{\text{top}} \to S_{\text{top}} + r^2 S \tag{1.10}$$

is equivalent to shifting the 't Hooft coupling constant of the Yang-Mills theory.

In addition to providing a string dual to free super-Yang-Mills, this topological string also describes an unbroken phase of closed superstring theory in which all background fields (including the metric) are treated on the same footing. Up to BRST-trivial terms, the topological action of (1.8) is independent of any specific choice for the spacetime metric, which was one of the original motivations of Witten for studying topological string theory [19–21]. To recover non-topological backgrounds, one gives expectation values to the physical moduli of the topological string. For example, the $AdS_5 \times S^5$ background at nonzero radius is obtained by perturbing with the physical vertex operator of (1.1) for the radius modulus, and other string theory backgrounds which are asymptotically $AdS_5 \times S^5$ can be obtained by perturbing with vertex operators corresponding to other physical moduli.

As in previous topological proposals of Witten for an unbroken phase of string theory, the target spacetime in the topological sigma model requires a complex structure [20, 21]. But unlike in previous proposals, the complex structure of spacetime is now dynamical and is determined by the pure spinor ghost variables λ^α and $\widehat{\lambda}^{\widehat{\alpha}}$ which choose a U(5) subgroup of (Wick-rotated) SO(10).[1] This can be seen from the kinetic term for the ten x's in the first term of (1.8) which, to quadratic order, is $\int d^2z(2\eta\lambda\widehat{\lambda})^{-1}(\lambda\gamma_a\gamma_b\widehat{\lambda})\partial x^a\overline{\partial}x^b$.

In section 2 of this paper, the pure spinor version of the $AdS_5 \times S^5$ sigma model will be reviewed. In section 3, it will be shown that non-minimal variables are unnecessary in this model, that the zero mode measure factor and b ghost are much simpler than in a flat background, and that a partial simplification also occurs in the Ramond-Ramond plane-wave background. In section 4, a BRST-trivial version of the $AdS_5 \times S^5$ sigma model will be constructed by gauge-fixing a G/G principal chiral model, and this topological model will be argued to describe the dual of free super-Yang-Mills. In section 5, conclusions and open problems will be discussed.

2 Review of $AdS_5 \times S^5$ sigma model

The pure spinor version of the worldsheet action for the $AdS_5 \times S^5$ superstring can be derived either by constructing the pure spinor action in a general curved background [23] and setting the background superfields to their $AdS_5 \times S^5$ values, or by adding terms to the Green-Schwarz $AdS_5 \times S^5$ action which replace κ symmetry with BRST invariance [24]. The second approach is more direct and will be reviewed here. The structure of supergravity vertex operators will then be discussed.

[1]Similar observations on pure spinors and topological strings have been made by N. Nekrasov [22].

2.1 Green-Schwarz worldsheet action

In a general Type II supergravity background, the Green-Schwarz action is

$$\int d^2z \frac{1}{2}(G_{MN}(Z)+B_{MN}(Z))\partial Z^M \overline{\partial} Z^N = \int d^2z \frac{1}{2}(\eta_{ab}E^a_M(Z)E^b_N(Z)+B_{MN}(Z))\partial Z^M \overline{\partial} Z^N \tag{2.1}$$

where $Z^M = (x^m, \theta^\mu, \widehat{\theta^{\widehat{\mu}}})$, $E^A_M(Z)$ is the super-vierbein, $A = (a, \alpha, \widehat{\alpha})$ are tangent-superspace variables for $a = 0$ to 9 and $\alpha, \widehat{\alpha} = 1$ to 16, and $M = (m, \mu, \widehat{\mu})$ are coordinate variables for $m = 0$ to 9 and $\mu, \widehat{\mu} = 1$ to 16, and (α, μ) and $(\widehat{\alpha}, \widehat{\mu})$ label spinors of the opposite/same chirality for the Type IIA/B superstring.

In an $AdS_5 \times S^5$ background, the supervierbein E^A_M can be explicitly constructed from the Metsaev-Tseytlin left-invariant currents $J^{\tilde{A}} = (g^{-1}\partial g)^{\tilde{A}}$ where g takes values in the coset PSU(2,2|4)/(SO(4,1) × SO(5)), $\tilde{A} = ([ab], a, \alpha, \widehat{\alpha})$ ranges over the 30 bosonic and 32 fermionic elements in the Lie algebra of PSU(2,2|4), $[ab]$ labels the SO(4,1) × SO(5) "Lorentz" generators, $a = 0$ to 9 labels the "translation" generators, and $\alpha, \widehat{\alpha} = 1$ to 16 label the fermionic "supersymmetry" generators. Note that $\tilde{A}$ includes both the superspace indices A as well as the SO(4,1) × SO(5) indices $[ab]$. The PSU(2,2|4) structure constants $f^{\tilde{C}}_{\tilde{A}\tilde{B}}$ include $f^a_{\alpha\beta} = \gamma^a_{\alpha\beta}$ and $f^a_{\widehat{\alpha}\widehat{\beta}} = \gamma^a_{\widehat{\alpha}\widehat{\beta}}$ where $\gamma^a_{\alpha\beta}$ and $(\gamma^a)^{\alpha\beta}$ are the 16 × 16 off-diagonal elements in the Weyl representation of the 32 × 32 ten-dimensional Γ-matrices, and $\gamma^a_{\widehat{\alpha}\widehat{\beta}}$ and $(\gamma^a)^{\widehat{\alpha}\widehat{\beta}}$ are related to these matrices by

$$\gamma^a_{\widehat{\alpha}\widehat{\beta}} \equiv \eta_{\alpha\widehat{\alpha}}\eta_{\beta\widehat{\beta}}(\gamma^a)^{\alpha\beta}, \quad (\gamma^a)^{\widehat{\alpha}\widehat{\beta}} \equiv \eta^{\alpha\widehat{\alpha}}\eta^{\beta\widehat{\beta}}\gamma^a_{\alpha\beta}, \quad \eta_{\alpha\widehat{\beta}} \equiv (\gamma^{01234})_{\alpha\widehat{\beta}}, \quad \eta^{\alpha\widehat{\beta}} \equiv (\gamma^{01234})^{\alpha\widehat{\beta}}. \tag{2.2}$$

Parameterizing the $AdS_5 \times S^5$ coset as

$$g(Z) = \exp(x^m P_m + \theta^\mu Q_\mu + \widehat{\theta}^{\widehat{\mu}} \widehat{Q}_{\widehat{\mu}}) \tag{2.3}$$

where $[P_m, Q_\mu, \widehat{Q}_{\widehat{\mu}}]$ are the $AdS_5 \times S^5$ translation and supersymmetry generators, one obtains

$$J^A = E^A_M(Z)\partial Z^M, \quad J^{[ab]} = \omega^{[ab]}_M(Z)\partial Z^M \tag{2.4}$$

where $\omega^{[ab]}_M$ is the $AdS_5 \times S^5$ spin connection. Furthermore, in an $AdS_5 \times S^5$ background, it was shown in [25] that the only nonzero components of $B_{AB} = E^M_A E^N_B B_{MN}$ are

$$B_{\alpha\widehat{\beta}} = B_{\widehat{\beta}\alpha} = \frac{1}{2}(\gamma^{01234})_{\alpha\widehat{\beta}} \equiv \frac{1}{2}\eta_{\alpha\widehat{\beta}}. \tag{2.5}$$

So the Green-Schwarz action in an $AdS_5 \times S^5$ background is [3, 25]

$$S_{\mathrm{GS}} = \int d^2z \left(\frac{1}{2}\eta_{ab}J^a\overline{J}^b + \frac{1}{4}\eta_{\alpha\widehat{\beta}}(J^\alpha \overline{J}^{\widehat{\beta}} - \overline{J}^\alpha J^{\widehat{\beta}}) \right). \tag{2.6}$$

Note that unlike the Green-Schwarz Lagrangian in a flat background in which the term $B_{MN}\partial Z^M \overline{\partial} Z^N$ transforms by a total derivative under spacetime supersymmetry, the Green-Schwarz Lagrangian in an $AdS_5 \times S^5$ background is manifestly PSU(2,2|4) invariant since it can be expressed in terms of the supersymmetric invariants J^A.

JHEP09(2009)051

2.2 Pure spinor worldsheet action

To generalize the Green-Schwarz action to the pure spinor formalism, one needs to add canonical momenta $(d_\alpha, \widehat{d}_{\widehat{\alpha}})$ for the $(\theta^\mu, \widehat{\theta}^{\widehat{\mu}})$ variables as well as left and right-moving pure spinor ghosts, $(\lambda^\alpha, w_\alpha)$ and $(\widehat{\lambda}^{\widehat{\alpha}}, \widehat{w}_{\widehat{\alpha}})$, which satisfy the pure spinor constraints $\lambda\gamma^a\lambda = \widehat{\lambda}\gamma^a\widehat{\lambda} = 0$. Because of the pure spinor constraints, w_α and $\widehat{w}_{\widehat{\alpha}}$ can only appear in combinations which are invariant under the gauge transformations

$$\delta w_\alpha = \xi^a(\gamma_a\lambda)_\alpha, \quad \delta\widehat{w}_{\widehat{\alpha}} = \widehat{\xi}^a(\gamma_a\widehat{\lambda})_{\widehat{\alpha}}, \tag{2.7}$$

which implies that they only appear through the Lorentz currents and ghost currents

$$N_{ab} = \frac{1}{2}w\gamma_{ab}\lambda, \quad J_{gh} = w_\alpha\lambda^\alpha, \qquad \widehat{N}_{ab} = \frac{1}{2}\widehat{w}\gamma_{ab}\widehat{\lambda}, \quad \widehat{J}_{gh} = \widehat{w}_{\widehat{\alpha}}\widehat{\lambda}^{\widehat{\alpha}}. \tag{2.8}$$

In an $AdS_5 \times S^5$ background, these additional worldsheet fields couple as

$$S = S_{\rm GS} + \int d^2z[-d_\alpha\overline{J}^\alpha + \widehat{d}_{\widehat{\alpha}}J^{\widehat{\alpha}} + d_\alpha\widehat{d}_{\widehat{\beta}}F^{\alpha\widehat{\beta}} - w_\alpha(\overline{\nabla}\lambda)^\alpha + \widehat{w}_{\widehat{\alpha}}(\nabla\widehat{\lambda})^{\widehat{\alpha}} + R_{abcd}N^{ab}\widehat{N}^{cd}] \tag{2.9}$$

where $F^{\alpha\widehat{\beta}} = (\gamma_{01234})^{\alpha\widehat{\beta}} \equiv \eta^{\alpha\widehat{\beta}}$ is the bispinor Ramond-Ramond field-strength, $R_{abcd} = \mp\eta_{a[c}\eta_{d]b} \equiv -\eta_{[ab][cd]}$ is the $AdS_5 \times S^5$ curvature (the $-$ sign is if a,b,c,d are on AdS_5 and the $+$ sign is if they are on S^5), and

$$(\overline{\nabla}\lambda)^\alpha = \overline{\partial}\lambda^\alpha + \frac{1}{2}\overline{J}^{[ab]}(\gamma_{ab}\lambda)^\alpha, \quad (\nabla\widehat{\lambda})^{\widehat{\alpha}} = \partial\widehat{\lambda}^{\widehat{\alpha}} + \frac{1}{2}J^{[ab]}(\gamma_{ab}\widehat{\lambda})^{\widehat{\alpha}}. \tag{2.10}$$

Because of the nonvanishing Ramond-Ramond flux, d_α and $\widehat{d}_{\widehat{\alpha}}$ are auxiliary fields which can be integrated out to give the action

$$S = \int d^2z\Bigg[\frac{1}{2}\eta_{ab}J^a\overline{J}^b - \eta_{\alpha\widehat{\beta}}\left(\frac{3}{4}J^{\widehat{\beta}}\overline{J}^\alpha + \frac{1}{4}\overline{J}^{\widehat{\beta}}J^\alpha\right) \tag{2.11}$$
$$\qquad -w_\alpha\overline{\nabla}\lambda^\alpha + \widehat{w}_{\widehat{\alpha}}\nabla\widehat{\lambda}^{\widehat{\alpha}} - \eta_{[ab][cd]}N^{ab}\widehat{N}^{cd}\Bigg]$$
$$= \int d^2z\Bigg[\frac{1}{2}\left(\eta_{ab}J^a\overline{J}^b + \eta_{\alpha\widehat{\beta}}J^\alpha\overline{J}^{\widehat{\beta}} + \eta_{\alpha\widehat{\beta}}\overline{J}^\alpha J^{\widehat{\beta}}\right) - \frac{1}{4}\eta_{\alpha\widehat{\beta}}(J^\alpha\overline{J}^{\widehat{\beta}} - \overline{J}^\alpha J^{\widehat{\beta}}) \tag{2.12}$$
$$\qquad +(-w_\alpha\overline{\nabla}\lambda^\alpha + \widehat{w}_{\widehat{\alpha}}\nabla\widehat{\lambda}^{\widehat{\alpha}} - \eta_{[ab][cd]}N^{ab}\widehat{N}^{cd})\Bigg].$$

The action of (2.11) is manifestly invariant under global PSU(2,2|4) transformations which transform $g(x,\theta,\widehat{\theta})$ by left multiplication as $\delta g = (\Sigma^{\tilde{A}}T_{\tilde{A}})g$ where $T_{\tilde{A}}$ are the PSU(2,2|4) Lie-algebra generators and is also manifestly invariant under local SO(4,1) $\times$ SO(5) gauge transformations which transform $g(x,\theta,\widehat{\theta})$ by right multiplication as $\delta_\Lambda g = g(\Lambda^{[ab]}T_{[ab]})$ and transform the pure spinors as SO(4,1) $\times$ SO(5) target-space spinors.

The BRST operator in the pure spinor formalism is defined as

$$Q = \int dz\ \lambda^\alpha d_\alpha + \int d\overline{z}\ \widehat{\lambda}^{\widehat{\alpha}}\widehat{d}_{\widehat{\alpha}} = \int dz\ \eta_{\alpha\widehat{\alpha}}\lambda^\alpha J^{\widehat{\alpha}} + \int d\overline{z}\ \eta_{\alpha\widehat{\alpha}}\widehat{\lambda}^{\widehat{\alpha}}\overline{J}^\alpha, \tag{2.13}$$

where the auxiliary equations of motion for d_α and $\widehat{d}_{\widehat{\alpha}}$ have been used. Under BRST transformations generated by Q, $g(x,\theta,\widehat{\theta})$ transforms by right-multiplication as

$$Q(g) = g(\lambda^\alpha T_\alpha + \widehat{\lambda}^{\widehat{\alpha}} T_{\widehat{\alpha}}) \tag{2.14}$$

which implies that

$$QJ^\alpha = \nabla\lambda^\alpha - \eta^{\alpha\widehat{\alpha}}(\gamma_a\widehat{\lambda})_{\widehat{\alpha}}J^a, \qquad QJ^{\widehat{\alpha}} = \nabla\widehat{\lambda}^{\widehat{\alpha}} + \eta^{\alpha\widehat{\alpha}}(\gamma_a\lambda)_\alpha J^a, \tag{2.15}$$

$$QJ^a = (\gamma_a\lambda)_\alpha J^\alpha + (\gamma_a\widehat{\lambda})_{\widehat{\alpha}}J^{\widehat{\alpha}}, \quad QJ^{[ab]} = \frac{1}{2}\eta^{[ab][cd]}\eta_{\alpha\widehat{\alpha}}(J^{\widehat{\alpha}}(\gamma_{cd}\lambda)^\alpha - J^\alpha(\gamma_{cd}\widehat{\lambda})^{\widehat{\alpha}}). \tag{2.16}$$

And (2.13) implies that the pure spinors transform as

$$Q(w_\alpha) = \eta_{\alpha\widehat{\alpha}}J^{\widehat{\alpha}}, \quad Q(\widehat{w}_{\widehat{\alpha}}) = \eta_{\alpha\widehat{\alpha}}\overline{J}^\alpha, \quad Q(\lambda^\alpha) = Q(\widehat{\lambda}^{\widehat{\alpha}}) = 0. \tag{2.17}$$

To verify that (2.11) is BRST invariant, note that the first term in the Lagrangian of (2.12) transforms under (2.13) to

$$\frac{1}{2}\eta_{\alpha\widehat{\alpha}}(J^{\widehat{\alpha}}\overline{\nabla}\lambda^\alpha + \overline{J}^{\widehat{\alpha}}\nabla\lambda^\alpha - J^\alpha\overline{\nabla}\widehat{\lambda}^{\widehat{\alpha}} - \overline{J}^\alpha\nabla\widehat{\lambda}^{\widehat{\alpha}}).$$

Using the Maurer-Cartan equations

$$\nabla\overline{J}^{\widehat{\alpha}} - \overline{\nabla}J^{\widehat{\alpha}} = \gamma_a^{\widehat{\alpha}\widehat{\beta}}\eta_{\beta\widehat{\beta}}(J^\beta\overline{J}^a - \overline{J}^\beta J^a), \quad \nabla\overline{J}^\alpha - \overline{\nabla}J^\alpha = -\gamma_a^{\alpha\beta}\eta_{\beta\widehat{\beta}}(J^{\widehat{\beta}}\overline{J}^a - \overline{J}^{\widehat{\beta}}J^a), \tag{2.18}$$

the second term in (2.12) transforms under (2.13) to

$$\begin{aligned}&\frac{1}{2}\eta_{\alpha\widehat{\alpha}}(J^{\widehat{\alpha}}\overline{\nabla}\lambda^\alpha - \overline{J}^{\widehat{\alpha}}\nabla\lambda^\alpha + J^\alpha\overline{\nabla}\widehat{\lambda}^{\widehat{\alpha}} - \overline{J}^\alpha\nabla\widehat{\lambda}^{\widehat{\alpha}}) \\ &+\frac{1}{4}\eta_{\alpha\widehat{\alpha}}\partial(\overline{J}^{\widehat{\alpha}}\lambda^\alpha + \overline{J}^\alpha\widehat{\lambda}^{\widehat{\alpha}}) - \frac{1}{4}\eta_{\alpha\widehat{\alpha}}\overline{\partial}(J^{\widehat{\alpha}}\lambda^\alpha + J^\alpha\widehat{\lambda}^{\widehat{\alpha}}).\end{aligned} \tag{2.19}$$

And the last term in (2.12) transforms under (2.13) to

$$-\eta_{\alpha\widehat{\alpha}}(J^{\widehat{\alpha}}\overline{\nabla}\lambda^\alpha - \overline{J}^\alpha\nabla\widehat{\lambda}^{\widehat{\alpha}}).$$

So ignoring the total derivatives in the second line of (2.19), (2.11) is BRST-invariant.

2.3 Nilpotent BRST transformations

Although it is consistent to use the BRST transformations of (2.14) and (2.17) which are nilpotent up to equations of motion, it will be convenient to include auxiliary antifields in the action so that the BRST transformations become nilpotent without using equations of motion. As discussed in [10] and shown independently by G. Boussard [26], this is easily done by adding the antifields w^*_α and $\widehat{w}^*_{\widehat{\alpha}}$ to the $AdS_5 \times S^5$ action of (2.11) as

$$S \to S + \int d^2z\,\eta^{\alpha\widehat{\alpha}}w^*_\alpha\widehat{w}^*_{\widehat{\alpha}} \tag{2.20}$$

where w^*_α and $\widehat{w}^*_{\widehat{\alpha}}$ are auxiliary fermionic spinors which are constrained to satisfy

$$\eta_{\alpha\widehat{\alpha}}(w^*\gamma^a)^\alpha\widehat{\lambda}^{\widehat{\alpha}} = 0, \quad \eta_{\alpha\widehat{\alpha}}(\widehat{w}^*\gamma^a)^{\widehat{\alpha}}\lambda^\alpha = 0, \tag{2.21}$$

JHEP09(2009)051

and therefore each contain 11 independent fermionic components.

Under the BRST transformations of (2.14) and (2.17), one finds that

$$Q^2 g = -g(h^{[ab]}T_{[ab]}), \tag{2.22}$$

$$Q^2 w_\alpha = \frac{1}{2}(\gamma_{ab} w)_\alpha h^{[ab]} + (\lambda\gamma_a)_\alpha \xi^a + \eta_{\alpha\widehat{\alpha}} \frac{\partial L}{\partial \widehat{w}_{\widehat{\alpha}}}, \tag{2.23}$$

$$Q^2 \widehat{w}_{\widehat{\alpha}} = \frac{1}{2}(\gamma_{ab}\widehat{w})_{\widehat{\alpha}} h^{[ab]} + (\widehat{\lambda}\gamma_a)_{\widehat{\alpha}} \widehat{\xi}^a - \eta_{\alpha\widehat{\alpha}} \frac{\partial L}{\partial w_\alpha}, \tag{2.24}$$

where

$$h^{[ab]} = \frac{1}{2}\eta_{\alpha\widehat{\alpha}}\lambda^\alpha(\gamma^{ab}\widehat{\lambda})^{\widehat{\alpha}}, \quad \xi^a = J^a - \eta^{\alpha\widehat{\alpha}} w_\alpha (\gamma^a\widehat{\lambda})_{\widehat{\alpha}}, \quad \widehat{\xi}^a = -\overline{J}^a + \eta^{\alpha\widehat{\alpha}}\widehat{w}_{\widehat{\alpha}}(\gamma^a\lambda)_\alpha, \tag{2.25}$$

$$\frac{\partial L}{\partial \widehat{w}_{\widehat{\alpha}}} = \nabla\widehat{\lambda}^{\widehat{\alpha}} - \frac{1}{2}\eta_{[ab][cd]} N^{ab}(\gamma^{cd}\widehat{\lambda})^{\widehat{\alpha}}, \qquad \frac{\partial L}{\partial w_\alpha} = -\overline{\nabla}\lambda^\alpha - \frac{1}{2}\eta_{[ab][cd]}(\gamma^{ab}\lambda)^\alpha \widehat{N}^{cd}. \tag{2.26}$$

When acting on terms which are gauge-invariant with respect to the local SO(4,1) × SO(5) transformations and the $(w, \widehat{w})$ gauge transformations of (2.7), the terms in (2.22) which are proportional to $(h^{[ab]}, \xi^a, \widehat{\xi}^a)$ can be ignored. To remove the terms in (2.22) which are proportional to the equations of motion $\frac{\partial L}{\partial w_\alpha}$ and $\frac{\partial L}{\partial \widehat{w}_{\widehat{\alpha}}}$, one should modify the BRST transformations of w_α and $\widehat{w}_{\widehat{\alpha}}$ to

$$Q w_\alpha = \eta_{\alpha\widehat{\alpha}} J^{\widehat{\alpha}} + w^*_\alpha, \quad Q\widehat{w}_{\widehat{\alpha}} = \eta_{\alpha\widehat{\alpha}} \overline{J}^\alpha + \widehat{w}^*_{\widehat{\alpha}}, \tag{2.27}$$

and define the BRST transformation of the antifields w^*_α and $\widehat{w}^*_{\widehat{\alpha}}$ as

$$Q w^*_\alpha = -\eta_{\alpha\widehat{\alpha}} \frac{\partial L}{\partial \widehat{w}_{\widehat{\alpha}}}, \quad Q\widehat{w}^*_{\widehat{\alpha}} = \eta_{\alpha\widehat{\alpha}} \frac{\partial L}{\partial w_\alpha}.$$

With the addition of (2.20) to the action, one can easily check that these BRST transformation leave the action invariant and are nilpotent without using equations of motion.

2.4 Supergravity vertex operators

In a general curved supergravity background, physical closed string vertex operators in the pure spinor formalism are defined as states of ghost-number (1,1) which are in the BRST cohomology. For massless supergravity states, these vertex operators only depend on the zero modes of the worldsheet fields $Z^M = (x^m, \theta^\mu, \widehat{\theta}^{\widehat{\mu}})$ as

$$V = \lambda^\alpha \widehat{\lambda}^{\widehat{\alpha}} A_{\alpha\widehat{\alpha}}(Z^M). \tag{2.28}$$

Under the BRST transformation generated by $Q = \int dz \lambda^\alpha d_\alpha + \int d\overline{z} \widehat{\lambda}^{\widehat{\alpha}} \widehat{d}_{\widehat{\alpha}}$,

$$QZ^M = \lambda^\alpha E^M_\alpha(Z) + \widehat{\lambda}^{\widehat{\alpha}} E^M_{\widehat{\alpha}}(Z) \tag{2.29}$$

where E^M_A is the inverse supervierbein. So

$$QV = \lambda^\alpha\widehat{\lambda}^{\widehat{\alpha}}(\lambda^\beta E^M_\beta + \widehat{\lambda}^{\widehat{\beta}} E^M_{\widehat{\beta}})\partial_M A_{\alpha\widehat{\alpha}} = (\lambda^\beta\nabla_\beta + \widehat{\lambda}^{\widehat{\beta}}\nabla_{\widehat{\beta}})(\lambda^\alpha\widehat{\lambda}^{\widehat{\alpha}} A_{\alpha\widehat{\alpha}}) \tag{2.30}$$

where $\nabla_A = E_A^M(\partial_M + \omega_M^{[ab]} M_{[ab]})$ is the covariant derivative and $M^{[ab]}$ are tangent-space Lorentz generators which act on the spinor indices α and $\widehat{\alpha}$. Since $\lambda\gamma^a\lambda = \widehat{\lambda}\gamma^a\widehat{\lambda} = 0$, $QV = 0$ implies that $A_{\alpha\widehat{\alpha}}(Z)$ satisfies [27]

$$\gamma_{abcde}^{\alpha\gamma}\nabla_\gamma A_{\alpha\widehat{\beta}} = \gamma_{abcde}^{\widehat{\beta}\widehat{\gamma}}\nabla_{\widehat{\gamma}} A_{\alpha\widehat{\beta}} = 0 \tag{2.31}$$

for any choice of $[abcde]$. And the gauge transformation

$$\delta V = Q(\lambda^\alpha\Omega_\alpha + \widehat{\lambda}^{\widehat{\alpha}}\Omega_{\widehat{\alpha}}) = (\lambda^\beta\nabla_\beta + \widehat{\lambda}^{\widehat{\beta}}\nabla_{\widehat{\beta}})(\lambda^\alpha\Omega_\alpha + \widehat{\lambda}^{\widehat{\alpha}}\Omega_{\widehat{\alpha}}) \tag{2.32}$$

implies that $A_{\alpha\widehat{\alpha}}(Z)$ is defined up to the gauge transformation

$$\delta A_{\alpha\widehat{\alpha}} = \nabla_\alpha\Omega_{\widehat{\alpha}} + \nabla_{\widehat{\alpha}}\Omega_\alpha \tag{2.33}$$

where Ω_α and $\Omega_{\widehat{\alpha}}$ are restricted to satisfy

$$\gamma_{abcde}^{\alpha\beta}\nabla_\beta\Omega_\alpha = \gamma_{abcde}^{\widehat{\alpha}\widehat{\beta}}\nabla_{\widehat{\beta}}\Omega_{\widehat{\alpha}} = 0 \tag{2.34}$$

for any choice of $[abcde]$.

As shown in [23], these equations of motion and gauge invariances describe an onshell Type II supergravity multiplet. In terms of the standard supergravity superfields, $A_{\alpha\widehat{\alpha}}(Z)$ is identified with the spinor-spinor component $B_{\alpha\widehat{\beta}}$ of the two-form $B_{AB} = E_A^M E_B^N B_{MN}$ in the gauge where $(\gamma_{abcde})^{\alpha\beta}B_{\alpha\beta} = (\gamma_{abcde})^{\widehat{\alpha}\widehat{\beta}}B_{\widehat{\alpha}\widehat{\beta}} = 0$. The equations of motion of (2.31) follow from the superfield constraints

$$H_{\alpha\widehat{\beta}\widehat{\gamma}} = H_{\widehat{\alpha}\beta\gamma} = 0, \quad (\gamma_{abcde})^{\alpha\beta}T^D_{\alpha\beta} = (\gamma_{abcde})^{\widehat{\alpha}\widehat{\beta}}T^D_{\widehat{\alpha}\widehat{\beta}} = T^D_{\alpha\widehat{\alpha}} = 0, \tag{2.35}$$

where

$$H_{ABC} = E_A^M E_B^N E_C^P \partial_{[M}B_{NP)} = \nabla_{[A}B_{BC)} + T^D_{[AB}B_{C)D} \tag{2.36}$$

is the three-form field strength and T^D_{AB} is the superspace torsion. And the gauge transformations of (2.33) follow from the gauge transformations $\delta B_{MN} = \partial_{[M}\Omega_{N)}$ which imply that $\delta B_{AB} = \nabla_{[A}\Omega_{B)} + T^C_{AB}\Omega_C$.

In a flat background, the constraints of (2.31) can be easily solved in terms of plane-wave solutions as $A_{\alpha\widehat{\beta}}(Z) = A_{\alpha\widehat{\beta}}(k, \theta, \widehat{\theta})e^{ikx}$ where $k^2 = 0$. Furthermore, the holomorphic structure of the sigma model implies that $A_{\alpha\widehat{\alpha}}(k, \theta, \widehat{\theta})$ factorizes into $A_{\alpha\widehat{\alpha}}(k, \theta, \widehat{\theta}) = A_\alpha(k, \theta)A_{\widehat{\alpha}}(k, \widehat{\theta})$ where $A_\alpha(k, \theta)$ is the super-Yang-Mills spinor gauge field satisfying $(\gamma_{abcde})^{\alpha\beta}D_\alpha A_\beta = 0$ with $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + k_m\gamma^m_{\alpha\beta}\theta^\beta$.

Unfortunately, the non-holomorphic structure of the $AdS_5 \times S^5$ sigma model does not allow a similar factorization for $A_{\alpha\widehat{\beta}}(Z)$ in an $AdS_5 \times S^5$ background. Nevertheless, the fact that $B_{\alpha\widehat{\alpha}}$ has the background value of $\eta_{\alpha\widehat{\alpha}}$ in this background implies that the $\theta = \widehat{\theta} = 0$ component of $\eta^{\alpha\widehat{\alpha}}A_{\alpha\widehat{\alpha}}(Z)$ is the dilaton. The other components of $A_{\alpha\widehat{\alpha}}(Z)$ can be determined by acting with supersymmetry on the dilaton.

3 Simplifying the $AdS_5 \times S^5$ formalism

In this section, it will be explained that since $(\eta_{\alpha\widehat{\alpha}}\lambda^\alpha\widehat{\lambda}^{\widehat{\alpha}})$ is in the BRST cohomology in an $AdS_5 \times S^5$ background, there is no need to introduce the non-minimal variables which are necessary in a flat background to regularize the functional integral over the pure spinors. This simplifies the zero mode measure factor and b ghost in an $AdS_5 \times S^5$ background, and a partial simplification will also occur in the Ramond-Ramond plane-wave background.

3.1 BRST cohomology and extended Hilbert space

To show that $(\eta\lambda\widehat{\lambda})$ is in the BRST cohomology in an $AdS_5 \times S^5$ background, note that the surface term in (2.19) implies that

$$QL_{\mathrm{AdS}} = \partial\overline{f} - \overline{\partial}f \tag{3.1}$$

where L_{AdS} is the Lagrangian of (2.11) and

$$f = \frac{1}{4}\eta_{\alpha\widehat{\alpha}}(\lambda^\alpha J^{\widehat{\alpha}} + \widehat{\lambda}^{\widehat{\alpha}} J^\alpha), \quad \overline{f} = \frac{1}{4}\eta_{\alpha\widehat{\alpha}}(\lambda^\alpha \overline{J}^{\widehat{\alpha}} + \widehat{\lambda}^{\widehat{\alpha}} \overline{J}^\alpha). \tag{3.2}$$

Furthermore, since the BRST transformations of (2.14) and (2.27) are nilpotent, (3.1) implies that $Qf = \partial V$ and $Q\overline{f} = \overline{\partial}V$ for some V. One can easily check for f and $\overline{f}$ of (3.2) that $V = \frac{1}{4}\eta_{\alpha\widehat{\alpha}}\lambda^\alpha\lambda^{\widehat{\alpha}}$.

Since this procedure relates dimension $(1,1)$ integrated vertex operators and dimension $(0,0)$ unintegrated vertex operators, $V = (\eta\lambda\widehat{\lambda})$ is the unintegrated vertex operator associated with the $AdS_5 \times S^5$ Lagrangian. And since the AdS_5 radius which multiplies the Lagrangian is a physical modulus, $(\eta\lambda\widehat{\lambda})$ must be in the BRST cohomology. Note that in a flat background, the analogous procedure using the flat worldsheet Lagrangian produces the physical unintegrated vertex operator $V = (\lambda\gamma^m\theta)(\widehat{\lambda}\gamma_m\widehat{\theta})$.

Since $(\eta\lambda\widehat{\lambda})$ is in the BRST cohomology, it is consistent to impose the constraint that $(\eta\lambda\widehat{\lambda})$ is non-vanishing. If λ^α and $\eta_{\alpha\widehat{\alpha}}\widehat{\lambda}^{\widehat{\alpha}}$ are interpreted as complex conjugates, this constraint implies that at least one component of λ^α must be nonzero. In the presence of this constraint, the Hilbert space can be extended to include states which depend on inverse powers of $(\eta\lambda\widehat{\lambda})$.

As mentioned in the introduction, such an extension of the Hilbert space in a flat background would trivialize the BRST cohomology since it would allow the state $W = (\eta\lambda\widehat{\lambda})^{-1}(\eta_{\beta\widehat{\beta}}\theta^\beta\widehat{\lambda}^{\widehat{\beta}})$ which satisfies $QW = 1$. But since $(\eta\lambda\widehat{\lambda})$ is not BRST-trivial, there is no such W satisfying $QW = 1$ that can be constructed in an $AdS_5 \times S^5$ background.

3.2 b ghost

Since $[Q,T] = 0$ where

$$T = \frac{1}{2}\eta_{ab}J^aJ^b + \eta_{\alpha\widehat{\alpha}}J^\alpha J^{\widehat{\alpha}} - w_\alpha\nabla\lambda^\alpha \tag{3.3}$$

is the left-moving stress tensor, one can ask if there exists an operator b satisfying $\{Q,b\} = T$. Before extending the Hilbert space to include inverse powers of $(\eta\lambda\widehat{\lambda})$, such an operator does not exist. This situation is analogous to the situation in a flat background where, before introducing non-minimal fields, one cannot construct an operator b satisfying $\{Q,b\} = T_{\rm flat}$ where $T_{\rm flat} = \frac{1}{2}\partial x^m\partial x_m - p_\alpha\partial\theta^\alpha - w_\alpha\partial\lambda^\alpha$.

However, after extending the Hilbert space to include inverse powers of $(\eta\lambda\widehat{\lambda})$, the b operator can be defined as

$$b = (\eta\lambda\widehat{\lambda})^{-1}\widehat{\lambda}^{\widehat{\alpha}}\left[\frac{1}{2}\gamma_{a\widehat{\alpha}\widehat{\beta}}J^aJ^{\widehat{\beta}} + \frac{1}{4}(\gamma_{ab})_{\widehat{\alpha}}{}^{\widehat{\beta}}\eta_{\beta\widehat{\beta}}N^{ab}J^\beta + \frac{1}{4}\eta_{\alpha\widehat{\alpha}}J_{gh}J^\alpha\right]. \tag{3.4}$$

Note that (3.4) resembles the first term of the b ghost in a flat background which is [28]

$$b_{\rm flat} = (\lambda^\alpha\overline{\lambda}_\alpha)^{-1}\overline{\lambda}_\alpha\left[\frac{1}{2}\gamma_m^{\alpha\beta}\Pi^m d_\beta + \frac{1}{4}(\gamma_{mn})_\beta{}^\alpha N^{mn}\partial\theta^\beta + \frac{1}{4}J_{gh}\partial\theta^\alpha\right] + \cdots \tag{3.5}$$

where $\overline{\lambda}_\alpha$ is a non-minimal field and ... includes terms with more complicated dependence on the non-minimal fields.

To show that $\{Q,b\} = T$, use (2.14) to compute that

$$\begin{aligned}
Qb =& (\eta\lambda\widehat{\lambda})^{-1}\Bigg[\frac{1}{2}(\eta\lambda\widehat{\lambda})\eta_{ab}J^aJ^b + \frac{1}{2}(\lambda\gamma_a)_\alpha J^\alpha(\widehat{\lambda}\gamma^a)_{\widehat{\alpha}}J^{\widehat{\alpha}} \\
&+\frac{1}{4}\widehat{\lambda}^{\widehat{\alpha}}(\gamma^{ab})_{\widehat{\alpha}}{}^{\widehat{\beta}}\eta_{\beta\widehat{\beta}}N_{ab}\nabla\lambda^\beta + \frac{1}{8}(J^{\widehat{\alpha}}(\gamma^{ab})_{\widehat{\alpha}}{}^{\widehat{\beta}}\eta_{\beta\widehat{\beta}}\lambda^\beta)(\widehat{\lambda}^{\widehat{\gamma}}(\gamma^{ab})_{\widehat{\gamma}}{}^{\widehat{\delta}}\eta_{\delta\widehat{\delta}}J^\delta) \\
&+\frac{1}{4}(\eta_{\alpha\widehat{\alpha}}\lambda^\alpha J^{\widehat{\alpha}})(\eta_{\beta\widehat{\beta}}\widehat{\lambda}^{\widehat{\beta}}J^\beta) + \frac{1}{4}(\eta_{\alpha\widehat{\alpha}}\widehat{\lambda}^{\widehat{\alpha}}\nabla\lambda^\alpha)J_{gh}\Bigg] \\
=& \frac{1}{2}\eta_{ab}J^aJ^b + \eta_{\alpha\widehat{\alpha}}J^\alpha J^{\widehat{\alpha}} - w_\alpha\nabla\lambda^\alpha
\end{aligned} \tag{3.6}$$

where the identity

$$\delta_\alpha^\delta\delta_\beta^\gamma = \frac{1}{2}(\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} - \frac{1}{8}(\gamma^{ab})_\alpha{}^\gamma(\gamma_{ab})_\beta{}^\delta - \frac{1}{4}\delta_\alpha^\gamma\delta_\beta^\delta \tag{3.7}$$

has been used and terms proportional to w^*_α have been dropped since they vanish onshell. Note that normal-ordering terms are being ignored, so one only needs to use (2.14) to derive (3.6). Furthermore, note that

$$-w_\alpha\nabla\lambda^\alpha = \frac{1}{4}(\eta\lambda\widehat{\lambda})^{-1}[\widehat{\lambda}^{\widehat{\alpha}}(\gamma^{ab})_{\widehat{\alpha}}{}^{\widehat{\beta}}\eta_{\beta\widehat{\beta}}N_{ab}\nabla\lambda^\beta + (\eta_{\alpha\widehat{\alpha}}\widehat{\lambda}^{\widehat{\alpha}}\nabla\lambda^\alpha)J_{gh}]$$

using (3.7) and $\lambda\gamma^a\nabla\lambda = 0$. One can similarly define $\overline{b}$ satisfying $\{Q,\overline{b}\} = \overline{T}$ where $\overline{T} = \frac{1}{2}\eta_{ab}\overline{J}^a\overline{J}^b + \eta_{\alpha\widehat{\alpha}}\overline{J}^\alpha\overline{J}^{\widehat{\alpha}} + \widehat{w}_{\widehat{\alpha}}\overline{\nabla}\widehat{\lambda}^{\widehat{\alpha}}$ and

$$\overline{b} = (\eta\lambda\widehat{\lambda})^{-1}\lambda^\alpha\left[-\frac{1}{2}\gamma_{a\alpha\beta}\overline{J}^a\overline{J}^\beta - \frac{1}{4}(\gamma_{ab})_\alpha{}^\beta\eta_{\beta\widehat{\beta}}\widehat{N}^{ab}\overline{J}^{\widehat{\beta}} - \frac{1}{4}\eta_{\alpha\widehat{\alpha}}\widehat{J}_{gh}\overline{J}^{\widehat{\alpha}}\right]. \tag{3.8}$$

Note that b is not holomorphic but $\overline{\partial}b$ is BRST-trivial. The g-loop amplitude prescription in the pure spinor formalism is given by

$$A_g = \int d^{3g-3}\tau\int d^{3g-3}\overline{\tau}\langle\left(\int\mu b\right)^{3g-3}\left(\int\overline{\mu}\overline{b}\right)^{3g-3}\prod_{r=1}^N\int d^2z_rU_r(z_r)\rangle \tag{3.9}$$

where U_r are the dimension $(1,1)$ integrated vertex operators and μ and $\overline{\mu}$ are the Beltrami differentials associated with the Teichmuller parameters τ and $\overline{\tau}$. One normally requires $\overline{\partial} b = 0$ so that $(\int \mu b)$ is invariant under transformations that shift μ by $\overline{\partial}\nu$ for any ν. However, assuming that BRST-trivial terms in the integrand do not contribute, it seems to be sufficient to only require that $\overline{\partial} b$ is BRST-trivial.

3.3 Functional integration and measure factor

In a flat background, functional integration over the 22 zero modes of λ^α and $\widehat{\lambda}^{\widehat{\alpha}}$ produces a divergent factor since these bosonic zero modes are non-compact. The most convenient method for regularizing this divergence is to introduce "non-minimal" variables $\overline{\lambda}_\alpha$ and $\overline{\widehat{\lambda}}_{\widehat{\alpha}}$, together with their BRST superpartners r_α and $\widehat{r}_{\widehat{\alpha}}$, and to modify the BRST operator to [4, 5, 29]

$$Q_{non-min} = \int dz(\lambda^\alpha d_\alpha + r_\alpha \overline{w}^\alpha) + \int d\overline{z}(\widehat{\lambda}^{\widehat{\alpha}} \widehat{d}_{\widehat{\alpha}} + \widehat{r}_{\widehat{\alpha}} \overline{\widehat{w}}^{\widehat{\alpha}}) \tag{3.10}$$

where $\overline{w}^\alpha$ and $\overline{\widehat{w}}^{\widehat{\alpha}}$ are the conjugate momenta for $\overline{\lambda}_\alpha$ and $\overline{\widehat{\lambda}}_{\widehat{\alpha}}$ and the non-minimal variables satisfy the constraints

$$\overline{\lambda}\gamma^m\overline{\lambda} = \overline{\lambda}\gamma^m r = \overline{\widehat{\lambda}}\gamma^m\overline{\widehat{\lambda}} = \overline{\widehat{\lambda}}\gamma^m\widehat{r} = 0. \tag{3.11}$$

One then inserts the regulator

$$\mathcal{N} = \exp[-\rho\, Q(\theta^\alpha\overline{\lambda}_\alpha + \widehat{\theta}^{\widehat{\alpha}}\overline{\widehat{\lambda}}_{\widehat{\alpha}})] = \exp[-\rho(\lambda^\alpha\overline{\lambda}_\alpha + \widehat{\lambda}^{\widehat{\alpha}}\overline{\widehat{\lambda}}_{\widehat{\alpha}} - \theta^\alpha r_\alpha - \widehat{\theta}^{\widehat{\alpha}}\widehat{r}_{\widehat{\alpha}})] \tag{3.12}$$

into the functional integral where ρ is a positive constant. Since $\mathcal{N}-1$ is BRST-trivial, the amplitude must be independent of the constant ρ and the location of $\mathcal{N}$. Treating $\overline{\lambda}_\alpha$ and $\overline{\widehat{\lambda}}_{\widehat{\alpha}}$ as the complex conjugates of λ^α and $\widehat{\lambda}^{\widehat{\alpha}}$, the insertion of $\mathcal{N}$ regularizes the functional integration over the pure spinor ghost zero modes because of its Gaussian dependence on λ. As shown in [4], functional integration using this regularization method in a flat background implies that

$$\begin{aligned} \langle f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda})\rangle &= \int d^{10}x \int d^{11}\lambda d^{11}\widehat{\lambda} d^{11}\overline{\lambda} d^{11}\overline{\widehat{\lambda}} \int d^{16}\theta d^{16}\widehat{\theta} d^{11} r d^{11}\widehat{r}\, \mathcal{N}\, f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda}) \\ &= \int d^{10}x \int (d^5\theta)_{\alpha_1\ldots\alpha_5}(d^5\widehat{\theta})_{\widehat{\alpha}_1\ldots\widehat{\alpha}_5} \end{aligned} \tag{3.13}$$

$$\left(\gamma^m\frac{\partial}{\partial\lambda}\right)^{\alpha_1}\left(\gamma^n\frac{\partial}{\partial\lambda}\right)^{\alpha_2}\left(\gamma^p\frac{\partial}{\partial\lambda}\right)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5}\left(\gamma^q\frac{\partial}{\partial\widehat{\lambda}}\right)^{\widehat{\alpha}_1}\left(\gamma^r\frac{\partial}{\partial\widehat{\lambda}}\right)^{\widehat{\alpha}_2}\left(\gamma^s\frac{\partial}{\partial\widehat{\lambda}}\right)^{\widehat{\alpha}_3}(\gamma_{qrs})^{\widehat{\alpha}_4\widehat{\alpha}_5}$$
$$f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda})_{\theta=\widehat{\theta}=0}$$

where $f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda})$ is assumed to have ghost-number $(3,3)$ and be independent of the non-minimal fields. Note that (3.11) implies that r_α and $\widehat{r}_{\widehat{\alpha}}$ each have 11 independent components, and integration over these components reduces the $\int d^{16}\theta d^{16}\widehat{\theta}$ integral to $\int d^5\theta d^5\widehat{\theta}$ because of the r_α and $\widehat{r}_{\widehat{\alpha}}$ dependence in $\mathcal{N}$.

In an $AdS_5 \times S^5$ background, the fact that $(\eta\lambda\widehat{\lambda})$ is in the BRST cohomology allows one to treat λ^α and $\eta_{\alpha\widehat{\alpha}}\widehat{\lambda}^{\widehat{\alpha}}$ as complex conjugates instead of introducing non-minimal variables. Although the zero mode integral $\int d^{11}\lambda d^{11}\widehat{\lambda}$ diverges because of the scale factor in λ, one can easily regularize this divergence by restricting the zero modes of λ^α and $\widehat{\lambda}^{\widehat{\alpha}}$ to satisfy $(\eta\lambda\widehat{\lambda}) = \Lambda$ for some positive constant Λ. Since $(\eta\lambda\widehat{\lambda})$ is BRST-invariant, this regularization preserves BRST invariance. Furthermore, since the ghost-number anomaly implies that genus g amplitudes violate ghost-number by $(3-3g, 3-3g)$, the dependence on Λ can be absorbed by shifting the string coupling constant from e^ϕ to $e^{\phi'} = \Lambda^{-\frac{3}{2}}e^\phi$. In other words, the factor of $e^{(2g-2)\phi'} = \Lambda^{3-3g}e^{(2g-2)\phi}$ at genus g includes the Λ dependence.

With this regularization, the zero mode integration for tree amplitudes simplifies to

$$\langle f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda})\rangle = \int d^{10}x \int d^{16}\theta d^{16}\widehat{\theta}\ sdet(E_M^A)\ \int d^{10}\lambda d^{10}\widehat{\lambda}\ f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda}) \tag{3.14}$$

where $sdet(E_M^A)$ is the superdeterminant of the $AdS_5 \times S^5$ supervierbein and $\int d^{10}\lambda d^{10}\widehat{\lambda}$ is an integral over the projective pure spinors which (after Wick rotation) parameterize the compact space $\frac{\mathrm{SO}(10)}{\mathrm{U}(5)}$. For example, for three-point supergravity tree amplitudes,

$$f = (\lambda^\alpha\widehat{\lambda}^{\widehat{\alpha}}A^{(1)}_{\alpha\widehat{\alpha}}(Z))(\lambda^\alpha\widehat{\lambda}^{\widehat{\alpha}}A^{(2)}_{\alpha\widehat{\alpha}}(Z))(\lambda^\alpha\widehat{\lambda}^{\widehat{\alpha}}A^{(3)}_{\alpha\widehat{\alpha}}(Z)) \tag{3.15}$$

where $\lambda^\alpha\widehat{\lambda}^{\widehat{\alpha}}A_{\alpha\widehat{\alpha}}(Z)$ is the supergravity vertex operator of (2.28). Integrating over the projective pure spinors gives

$$\int d^{10}\lambda \int d^{10}\widehat{\lambda}\ f = T^{((\alpha\beta\gamma))((\widehat{\alpha}\widehat{\beta}\widehat{\gamma}))}A^{(1)}_{\alpha\widehat{\alpha}}(Z)A^{(2)}_{\beta\widehat{\beta}}(Z)A^{(3)}_{\gamma\widehat{\gamma}}(Z) \tag{3.16}$$

where $T^{((\alpha\beta\gamma))((\widehat{\alpha}\widehat{\beta}\widehat{\gamma}))}$ is the constant tensor obtained by symmetrizing $\eta^{\alpha\widehat{\alpha}}\eta^{\beta\widehat{\beta}}\eta^{\gamma\widehat{\gamma}}$ with respect to $(\alpha\beta\gamma)$ and $(\widehat{\alpha}\widehat{\beta}\widehat{\gamma})$ and removing the gamma-matrix trace terms, i.e. removing the terms proportional to $\gamma_m^{\alpha\beta}$ or $\gamma_m^{\widehat{\alpha}\widehat{\beta}}$.

So the onshell three-point tree amplitude in an $AdS_5 \times S^5$ background is claimed to be

$$\int d^{10}x \int d^{16}\theta d^{16}\widehat{\theta}\ sdet(E_M^A)\ T^{((\alpha\beta\gamma))((\widehat{\alpha}\widehat{\beta}\widehat{\gamma}))}A^{(1)}_{\alpha\widehat{\alpha}}(Z)A^{(2)}_{\beta\widehat{\beta}}(Z)A^{(3)}_{\gamma\widehat{\gamma}}(Z). \tag{3.17}$$

It might seem surprising that the zero mode integration in an $AdS_5 \times S^5$ background selects the term in $(A_{\alpha\widehat{\alpha}})^3$ with 16 $(\theta\widehat{\theta})$'s whereas the zero mode integration in a flat background selects the term in $(A_{\alpha\widehat{\alpha}})^3$ with 5 $(\theta\widehat{\theta})$'s. However, note that three-point amplitudes in an $AdS_5 \times S^5$ background can be computed as a sum over N-point amplitudes in a flat background where $(N-3)$ of the vertex operators deform the flat background to $AdS_5 \times S^5$. If 11 of the extra vertex operators are Ramond-Ramond vertex operators containing the term $\int d^2z F^{\alpha\widehat{\alpha}}d_\alpha\widehat{d}_{\widehat{\alpha}}$, one could contract 11 $(\theta\widehat{\theta})$'s in $(A_{\alpha\widehat{\alpha}})^3$ with these vertex operators and convert the flat zero-mode measure factor into the $AdS_5 \times S^5$ measure factor. So the $\int d^2z F^{\alpha\widehat{\alpha}}d_\alpha d_{\widehat{\alpha}}$ term in the $AdS_5 \times S^5$ action of (2.9) plays the same role as the $\exp[\rho(\theta^\alpha r_\alpha + \widehat{\theta}^{\widehat{\alpha}}\widehat{r}_{\widehat{\alpha}})]$ term in the regulator of (3.12) which absorbs 11 $(\theta\widehat{\theta})$'s after integrating over $\int d^{11}r \int d^{11}\widehat{r}$.

A separate argument for the validity of the integration measure of (3.14) is that it is manifestly PSU$(2,2|4)$ invariant since it can be written as

$$\langle f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda})\rangle = \int Dg \int d^{10}\lambda d^{10}\widehat{\lambda}\; f(g,\lambda,\widehat{\lambda}) \tag{3.18}$$

where g is the $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times \mathrm{SO}(5)}$ coset and Dg is the corresponding Haar measure. For three-point supergravity amplitudes in an $AdS_5 \times S^5$ background, PSU$(2,2|4)$ invariance together with gauge invariance is expected to completely fix the amplitude up to an overall constant.

This is analogous to the statement that the three-point supergravity amplitude in a flat background is completely fixed by super-Poincaré invariance and gauge invariance. In a flat background, the expression $\int d^{10}x \int d^{16}\theta \int d^{16}\widehat{\theta}(\lambda\widehat{\lambda}A)^3$ would vanish by dimensional arguments since it carries 11 too many factors of momentum and since $k_r\cdot k_s = 0$ for on-shell three-point amplitudes. For this reason, the correct measure factor in a flat background involves an integration over only 5 $(\theta\widehat{\theta})$'s. But in an $AdS_5 \times S^5$ background, there is no such dimensional argument since the expression $\int d^{10}x \int d^{16}\theta \int d^{16}\widehat{\theta}(\lambda\widehat{\lambda}A)^3$ can depend on inverse powers of the AdS radius as $(r_{\mathrm{AdS}})^{-11}$. So assuming that (3.17) does not vanish for some unknown reason, PSU$(2,2|4)$ invariance implies that it must be proportional to the correct three-point supergravity amplitude in an $AdS_5 \times S^5$ background.

In some sense, the above definition of the integration measure for pure spinors in an $AdS_5 \times S^5$ background is more natural than the corresponding definition in a flat background. Since the left and right-moving pure spinors are complex variables, it is natural (on a two-dimensional Euclidean worldsheet) to identify the right-moving pure spinor as the complex conjugate of the left-moving pure spinor so that the action is real. But as explained above, this identification is insufficient in a flat background for defining a regularized path integral since $(\lambda\hat{\lambda})$ is BRST-trivial. So one is forced to introduce left and right-moving non-minimal variables to regularize the path integral. However, in an $AdS_5 \times S^5$ background, identification of the left and right-moving pure spinor variables as complex conjugates allows one to define functional integration in the standard way without requiring non-minimal variables.

For amplitudes at non-zero genus, the prescription in the pure spinor formalism is to insert $(3g-3)$ b and $\overline{b}$ ghosts and N integrated vertex operators into the functional integral as in (3.9). After integrating out the non-zero modes of the worldsheet fields, one needs to integrate over both the zero modes of $(x,\theta,\widehat{\theta},\lambda,\widehat{\lambda})$ and the g zero modes of the spin-one variables w_α and $\widehat{w}_{\widehat{\alpha}}$. In a flat background, integration over the zero modes of w_α and $\widehat{w}_{\widehat{\alpha}}$ produces divergences which are regularized by including the term [4, 5]

$$\begin{aligned}\exp&\left[\rho\, Q\left(\frac{1}{2}(\overline{\lambda}\gamma^{ab}s)N_{ab} + \frac{1}{2}(\overline{\widehat{\lambda}}\gamma^{ab}\widehat{s})\widehat{N}_{ab}\right)\right] \\ &= \exp\left[-\rho\left(N^{ab}\overline{N}_{ab} + \widehat{N}^{ab}\overline{\widehat{N}}_{ab} - \frac{1}{4}(\overline{\lambda}\gamma^{ab}s)(\lambda\gamma_{ab}d) - \frac{1}{4}(\overline{\widehat{\lambda}}\gamma^{ab}\widehat{s})(\widehat{\lambda}\gamma_{ab}\widehat{d})\right)\right]\end{aligned} \tag{3.19}$$

in the regulator $\mathcal{N}$ of (3.12) where $\overline{N}_{ab}$ and $\overline{\widehat{N}}_{ab}$ are the Lorentz currents for the non-minimal variables and $(s^\alpha, \widehat{s}^{\widehat{\alpha}})$ are the conjugate momenta for $(r_\alpha, \widehat{r}_{\widehat{\alpha}})$. However, in an $AdS_5 \times S^5$ background, the worldsheet action of (2.9) already contains $\exp(-N^{ab}\widehat{N}_{ab})$ dependence because of the $AdS_5 \times S^5$ curvature which couples the left and right-moving Lorentz currents. So the curvature of the $AdS_5 \times S^5$ background acts as a regulator for the $(w_\alpha, \widehat{w}_{\widehat{\alpha}})$ zero mode integration and eliminates the need for the non-minimal regulator $\mathcal{N}$ of (3.19).

It should be noted that because of the non-holomorphic structure of the sigma model, the measure factor for open string scattering amplitudes in $AdS_5 \times S^5$ will not be the "holomorphic square-root" of the closed string measure factor of (3.14). For example, for D_3 branes at the boundary of AdS_5, the boundary condition $\widehat{\lambda}^{\widehat{\alpha}} = (\gamma_{0123})^{\widehat{\alpha}}_{\beta}\lambda^\beta$ implies that $\lambda\gamma^{01234}\widehat{\lambda} = \lambda\gamma^4\lambda = 0$ because of the pure spinor constraint $\lambda\gamma^a\lambda = 0$. So one cannot impose that $(\eta\lambda\widehat{\lambda}) = 0$ on the D_3 brane boundary.

To regularize the functional integral over pure spinors in the presence of D_3 branes, one therefore needs to introduce the same non-minimal variables $(\overline{\lambda}_\alpha, r_\alpha)$ on the boundary as one would introduce in a flat background. After inserting the non-minimal regulator $\mathcal{N} = \exp[-\rho(\lambda^\alpha\overline{\lambda}_\alpha - \theta^\alpha r_\alpha)]$ on the boundary and integrating over the non-minimal fields, the zero mode measure factor for open string amplitudes will involve integration over only 5 θ's. This is expected since open string amplitudes on $AdS_5 \times S^5$ describe $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills amplitudes which, like $d = 10$ super-Yang-Mills amplitudes, are naturally expressed in pure spinor superspace as integrals over 5 θ's [30, 31].

3.4 Ramond-Ramond plane-wave background

It is instructive to compare the structure of the zero-mode measure factors in flat and $AdS_5 \times S^5$ backgrounds with the zero-mode measure factor in a Ramond-Ramond plane-wave background. The pure spinor action in this background was described in [6] and has the same structure as (2.9) except that the non-vanishing components of $F^{\alpha\widehat{\beta}}$ and R_{abcd} take the values

$$F^{\alpha\widehat{\beta}} = \frac{1}{240}F^{mnpqr}\gamma^{\alpha\widehat{\beta}}_{mnpqr} = (\gamma_{-1234})^{\alpha\widehat{\beta}}, \quad R_{+j+k} = \delta_{jk} \tag{3.20}$$

where $x^\pm = x^0 \pm x^9$ and $j = 1$ to 8 denote the transverse directions.

Splitting d_α and $\widehat{d}_{\widehat{\alpha}}$ into their SO(8) components as

$$d_A = (\gamma_+\gamma_- d)_A, \quad d_{A'} = (\gamma_-\gamma_+ d)_{A'}, \quad \widehat{d}_{\widehat{A}} = (\gamma_+\gamma_-\widehat{d})_{\widehat{A}}, \quad \widehat{d}_{\widehat{A}'} = (\gamma_-\gamma_+\widehat{d})_{\widehat{A}'} \tag{3.21}$$

where $A, A' = 1$ to 8, the term $d_\alpha F^{\alpha\widehat{\beta}}\widehat{d}_{\widehat{\beta}}$ in (2.9) implies that d_A and $\widehat{d}_{\widehat{A}}$ are auxiliary variables which can be integrated out. But the variables $d_{A'}$ and $\widehat{d}_{\widehat{A}'}$ are propagating and couple to $\theta^{A'} = (\gamma^-\gamma^+\theta)^{A'}$ and $\widehat{\theta}^{\widehat{A}'} = (\gamma^-\gamma^+\widehat{\theta})^{\widehat{A}'}$ through the first-order action

$$\int d^2z[d_{A'}\overline{\partial}\theta^{A'} + \widehat{d}_{\widehat{A}'}\partial\widehat{\theta}^{\widehat{A}'}]. \tag{3.22}$$

In this plane-wave background, the operator $\eta_{\alpha\widehat{\alpha}}\lambda^\alpha\lambda^{\widehat{\alpha}}$ of (1.3) is replaced by $\lambda\gamma_{+1234}\widehat{\lambda} = \eta_{A\widehat{A}}\lambda^A\widehat{\lambda}^{\widehat{A}}$ where $\eta_{A\widehat{A}} \equiv (\sigma^{1234})_{A\widehat{A}}$ is constructed from the SO(8) Pauli matrices $\sigma^j_{AA'}$ and

$$\lambda^A = (\gamma^+\gamma^-\lambda)^A, \quad \lambda^{A'} = (\gamma^-\gamma^+\lambda)^{A'}, \quad \widehat{\lambda}^{\widehat{A}} = (\gamma^+\gamma^-\widehat{\lambda})^{\widehat{A}}, \quad \widehat{\lambda}^{\widehat{A}'} = (\gamma^-\gamma^+\widehat{\lambda})^{\widehat{A}'}. \tag{3.23}$$

JHEP09(2009)051

Since $\eta_{A\widehat{A}}\lambda^A\widehat{\lambda}^{\widehat{A}}$ is in the BRST cohomology, one can treat $\eta_{A\widehat{A}}\widehat{\lambda}^{\widehat{A}}$ as the complex conjugate of λ^A and impose the constraint that $\eta_{A\widehat{A}}\lambda^A\widehat{\lambda}^{\widehat{A}}$ is non-vanishing.

This resolves the problem of functional integration over λ^A and $\widehat{\lambda}^{\widehat{A}}$, but one still needs to regularize the functional integration over the remaining components $\lambda^{A'}$ and $\widehat{\lambda}^{\widehat{A}'}$ which are SO(8) pure spinors since they satisfy the constraint

$$\lambda^{A'}\lambda^{A'} = \widehat{\lambda}^{\widehat{A}'}\widehat{\lambda}^{\widehat{A}'} = 0 \tag{3.24}$$

coming from the condition $\lambda\gamma^+\lambda = \widehat{\lambda}\gamma^+\widehat{\lambda} = 0$. This regularization can be performed by introducing non-minimal fields $\overline{\lambda}_{A'}$ and $\overline{\widehat{\lambda}}_{\widehat{A}'}$ and their BRST superpartners $r_{A'}$ and $\widehat{r}_{\widehat{A}'}$ which satisfy the constraints

$$\overline{\lambda}_{A'}\overline{\lambda}_{A'} = \overline{\lambda}_{A'}r_{A'} = \overline{\widehat{\lambda}}_{\widehat{A}'}\overline{\widehat{\lambda}}_{\widehat{A}'} = \overline{\widehat{\lambda}}_{\widehat{A}'}\widehat{r}_{\widehat{A}'} = 0. \tag{3.25}$$

One then adds the term $\int dz r_{A'}\overline{w}^{A'} + \int d\overline{z}\widehat{r}_{\widehat{A}'}\overline{\widehat{w}}^{\widehat{A}'}$ to the BRST operator and defines the non-minimal regulator as

$$\mathcal{N} = \exp\Big[-\rho\, Q\Big(\theta^{A'}\overline{\lambda}_{A'} + \widehat{\theta}^{\widehat{A}'}\overline{\widehat{\lambda}}_{\widehat{A}'}\Big)\Big] = \exp\Big[-\rho\Big(\lambda^{A'}\overline{\lambda}_{A'} - \theta^{A'}r_{A'} + \widehat{\lambda}^{\widehat{A}'}\overline{\widehat{\lambda}}_{\widehat{A}'} - \widehat{\theta}^{\widehat{A}'}\widehat{r}_{\widehat{A}'}\Big)\Big] \tag{3.26}$$

Since there are seven independent $r_{A'}$ and $\widehat{r}_{\widehat{A}'}$ variables, the zero mode integration in a plane-wave background is of the form

$$\begin{aligned}\langle f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda})\rangle &= \int d^{10}x \int d^{11}\lambda d^{11}\widehat{\lambda} d^7\overline{\lambda} d^7\overline{\widehat{\lambda}} \int d^{16}\theta d^{16}\widehat{\theta} d^7 r d^7\widehat{r}\, \mathcal{N}\; f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda}) \quad (3.27)\\ &= \int d^{10}x \int d^8\theta^A \int d^8\widehat{\theta}^{\widehat{A}} \int d\lambda d\widehat{\lambda} \;\; \int d\theta_{A'}\frac{\partial}{\partial\lambda^{A'}} \;\; \int d\widehat{\theta}_{\widehat{A}'}\frac{\partial}{\partial\widehat{\lambda}^{\widehat{A}'}} f(x,\theta,\lambda,\widehat{\theta},\widehat{\lambda})|_{\theta=\widehat{\theta}=0}\end{aligned}$$

where the integration $\int d\lambda d\widehat{\lambda}$ is over the projective part of λ^A and $\widehat{\lambda}^{\widehat{A}}$ (keeping $\eta_{A\widehat{A}}\lambda^A\widehat{\lambda}^{\widehat{A}}$ fixed). So instead of selecting the term in f with 5 $(\theta\widehat{\theta})$'s or 16 $(\theta\widehat{\theta})$'s, the zero mode measure factor in a plane-wave background selects the term in f with 9 $(\theta\widehat{\theta})$'s.

Although this result may seem strange, it is consistent with the expectation from light-cone gauge analysis. In light-cone gauge, the supergravity vertex operator in a plane-wave background depends only on the transverse zero modes and has the form [15]

$$\Phi = f(a_j^\dagger, s_A^\dagger)|0\rangle \tag{3.28}$$

where $a_j^\dagger$ and $s_A^\dagger$ are 8 bosonic and 8 fermionic operators constructed from the zero modes which "excite" the ground-state wavefunction $|0\rangle$ of the harmonic oscillator for the massive zero modes. In terms of the zero modes $(x^j, \theta^A, \widehat{\theta}^{\widehat{A}})$, the Lagrangian is

$$\frac{1}{2}\dot{x}^j\dot{x}^j + \frac{i}{2}k^+(\theta^A\dot{\theta}^A + \widehat{\theta}^{\widehat{A}}\dot{\widehat{\theta}}^{\widehat{A}}) - (k^+)^2\left(\frac{1}{2}x^j x^j + i\eta_{A\widehat{A}}\theta^A\widehat{\theta}^{\widehat{A}}\right) \tag{3.29}$$

and the ground-state wavefunction is

$$|0\rangle = |4\pi k^+|^{-2}\exp\left(-|k^+|\left(\frac{1}{2}x^j x^j + i\eta_{A\widehat{A}}\theta^A\widehat{\theta}^{\widehat{A}}\right)\right) \tag{3.30}$$

where k^+ is the P^+ momentum of the state.

In light-cone gauge, the measure factor $\langle\Phi_1|\Phi_2\rangle_{\rm LC}$ can be computed either by using the commutation relations of the operators in (3.28) or by evaluating the functional integral

$$\langle\Phi_1|\Phi_2\rangle_{\rm LC} = \int d^8x \int d^8\theta \int d^8\widehat\theta \;\Phi_1(x^j,\theta^A,\widehat\theta^{\widehat A})\;\Phi_2(x^j,\theta^A,\widehat\theta^{\widehat A}). \tag{3.31}$$

Note that $|0\rangle$ has a well-defined norm since

$$\langle 0|0\rangle_{\rm LC} = \int d^8x \int d^8\theta \int d^8\widehat\theta |4\pi k^+|^{-4} e^{-|k^+|(x^jx^j+2i\eta_{A\widehat A}\theta^A\widehat\theta^{\widehat A})} = 1. \tag{3.32}$$

The covariant measure factor of (3.27) can be compared with the light-cone measure factor of (3.31) using the relation that $\langle V_1|c_0\overline c_0|V_2\rangle$ should be proportional to $\langle\Phi_1|\Phi_2\rangle_{\rm LC}$ where V is the BRST-invariant vertex operator of ghost-number $(1,1)$ corresponding to the light-cone vertex operator Φ, and c_0 and $\overline c_0$ are operators satisfying $\{b_0,c_0\}=\{\overline b_0,\overline c_0\}=1$. The factors of c_0 and $\overline c_0$ come from BRST gauge-fixing and are necessary for the covariant measure factor to have ghost-number $(3,3)$.

In a plane-wave background, the BRST-invariant vertex operator corresponding to the light-cone field $\Phi(x^j,\theta^A,\widehat\theta^{\widehat A})$ is

$$V = \lambda^\alpha\widehat\lambda^{\widehat\alpha}A_{\alpha\widehat\alpha}(x,\theta,\widehat\theta) = (\eta_{A\widehat A}\lambda^A\widehat\lambda^{\widehat A})\Phi(x^j,\theta^A,\widehat\theta^{\widehat A})e^{ik^+x^-+ik^-x^+} + \cdots \tag{3.33}$$

where Φ is the light-cone superfield of (3.28) and $\ldots$ depends on $\theta^{A'}$ and $\widehat\theta^{\widehat A'}$ and is determined by BRST invariance. Furthermore, since the b and $\overline b$ ghosts in the pure spinor formalism have the term

$$b = (\overline\lambda_{A'}\lambda^{A'})^{-1}\partial x^+(\overline\lambda\gamma^- d) + \cdots, \quad \overline b = (\overline\lambda_{A'}\lambda^{A'})^{-1}\overline\partial x^+(\overline{\widehat\lambda}\gamma^-\widehat d) + \cdots, \tag{3.34}$$

one can define c_0 and $\overline c_0$ satisfying $\{b_0,c_0\}=\{\overline b_0,\overline c_0\}=1$ as

$$c_0 = [(\partial x^+)^{-1}\lambda^{A'}\theta^{A'}]_0 = (k^+)^{-1}\lambda^{A'}\theta^{A'}, \quad \overline c_0 = [(\overline\partial x^+)^{-1}\widehat\lambda^{\widehat A'}\widehat\theta^{\widehat A'}]_0 = (k^+)^{-1}\widehat\lambda^{\widehat A'}\widehat\theta^{\widehat A'}. \tag{3.35}$$

So the covariant measure factor of (3.27) implies that

$$\begin{aligned}\langle V_1|c_0\overline c_0|V_2\rangle &= \int d^{10}x\int d^8\theta^A\int d^8\widehat\theta^{\widehat A}\int d\lambda d\widehat\lambda \int d\theta_{A'}\frac{\partial}{\partial\lambda^{A'}}\int d\widehat\theta_{\widehat A'}\frac{\partial}{\partial\widehat\lambda^{\widehat A'}} \\ &\quad (\eta_{A\widehat A}\lambda^A\widehat\lambda^{\widehat A})^2\;\Phi_1\Phi_2\;(k^+)^{-2}(\lambda^{A'}\theta^{A'})(\widehat\lambda^{\widehat A'}\widehat\theta^{\widehat A'})e^{i(k_1^++k_2^+)x^-+i(k_1^-+k_2^-)x^+} \\ &= (k^+)^{-2}\delta(k_1^++k_2^+)\delta(k_1^-+k_2^-)\int d^8x\int d^8\theta^A\int d^8\widehat\theta^{\widehat A}\;\Phi_1\Phi_2,\end{aligned} \tag{3.36}$$

which is proportional to the light-cone measure factor $\langle\Phi_1|\Phi_2\rangle_{\rm LC}$ of (3.31).

So in a plane-wave background, the covariant measure factor involving integration over 9 $(\theta\widehat\theta)$'s is related to light-cone integration over 8 $(\theta\widehat\theta)$'s plus an additional integration over $\theta\widehat\theta$ coming from the $c_0\overline c_0$ term. In a flat background, the covariant measure factor of (3.13) involving integration over 5 $(\theta\widehat\theta)$'s can be similarly related to light-cone integration over 4 $(\theta\widehat\theta)$'s plus an integration over $\theta\widehat\theta$ coming from the $c_0\overline c_0$ term. In light-cone gauge in a flat background, the fermionic zero modes are massless and in order to construct normalizable wavefunctions, the SO(8) components θ^A and $\widehat{\theta}^A$ need to be split into U(4) components as $(\theta^I, \overline{\theta}_I)$ and $(\widehat{\theta}^{\widehat{I}}, \overline{\widehat{\theta}}_{\widehat{I}})$ for $I, \widehat{I} = 1$ to 4 [32]. The resulting light-cone wavefunction is a chiral superfield $\Phi(\theta^I, \widehat{\theta}^{\widehat{I}})$ satisfying the reality condition

$$D_I D_J \widehat{D}_{\widehat{I}} \widehat{D}_{\widehat{J}} \Phi = \frac{1}{4} \epsilon_{IJKL} \epsilon_{\widehat{I}\widehat{J}\widehat{K}\widehat{L}} \overline{D}^K \overline{D}^L \overline{\widehat{D}}^{\widehat{K}} \overline{\widehat{D}}^{\widehat{L}} \overline{\Phi}, \tag{3.37}$$

and the light-cone measure factor in a flat background is

$$\langle \Phi_1 | \Phi_2 \rangle_{\rm LC} = \int d^8 x \int d^4 \theta^I \int d^4 \widehat{\theta}^{\widehat{I}} \; \Phi_1 \Phi_2 \tag{3.38}$$

which involves an integration over only 4 $(\theta\widehat{\theta})$'s.

4 Topological $AdS_5 \times S^5$ sigma model

In this section, a BRST-trivial action will be constructed with the same BRST operator and stress-tensor as the $AdS_5 \times S^5$ action of (2.11), and will be shown to arise from gauge-fixing the G/G principal chiral model where $G = \mathrm{PSU}(2,2|4)$. This topological action will then be argued to describe the zero-radius limit of $AdS_5 \times S^5$ by comparing its physical states with the spectrum of gauge-invariant operators of free $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills. A handwaving argument based on open-closed topological duality will then be proposed for showing that the scattering amplitudes of this topological string coincide with super-Yang-Mills scattering amplitudes in the limit of small 't Hooft coupling constant.

4.1 Topological action

Because of the possibility of including $(\eta\lambda\widehat{\lambda})^{-1}$ dependence in the action, one can construct a BRST-trivial action which has the same stress tensor as the $AdS_5 \times S^5$ action of (2.11). This topological action is

$$\begin{aligned} S_{\rm top} &= \int d^2 z \; Q(\Psi) \qquad\qquad (4.1) \\ &= \int d^2 z \left[\frac{\eta^{\alpha\widehat{\alpha}} (\gamma_a \lambda)_\alpha (\gamma_b \widehat{\lambda})_{\widehat{\alpha}}}{2(\eta\lambda\widehat{\lambda})} J^a \overline{J}^b + \eta_{\alpha\widehat{\alpha}} \overline{J}^\alpha J^{\widehat{\alpha}} - w_\alpha \overline{\nabla} \lambda^\alpha + \widehat{w}_{\widehat{\alpha}} \nabla \widehat{\lambda}^{\widehat{\alpha}} - \eta_{[ab][cd]} N^{ab} \widehat{N}^{cd} + \eta^{\alpha\widehat{\alpha}} w^*_\alpha \widehat{w}^*_{\widehat{\alpha}} \right] \end{aligned}$$

where

$$\begin{aligned} \Psi = & \frac{1}{2} (\eta\lambda\widehat{\lambda})^{-1} \widehat{\lambda}^{\widehat{\alpha}} \left(\frac{1}{2} \gamma_{a\widehat{\alpha}\widehat{\beta}} \overline{J}^a J^{\widehat{\beta}} + \frac{1}{4} (\gamma_{ab})_{\widehat{\alpha}}^{\widehat{\beta}} \eta_{\beta\widehat{\beta}} N^{ab} \overline{J}^\beta + \frac{1}{4} \eta_{\alpha\widehat{\alpha}} J_{gh} \overline{J}^\alpha \right) \\ & + \frac{1}{2} (\eta\lambda\widehat{\lambda})^{-1} \lambda^\alpha \left(- \frac{1}{2} \gamma_{a\alpha\beta} J^a \overline{J}^\beta - \frac{1}{4} (\gamma_{ab})_\alpha^\beta \eta_{\beta\widehat{\beta}} \widehat{N}^{ab} J^{\widehat{\beta}} - \frac{1}{4} \eta_{\alpha\widehat{\alpha}} \widehat{J}_{gh} J^{\widehat{\alpha}} \right) \\ & + \frac{1}{2} \eta^{\alpha\widehat{\alpha}} (w_\alpha \widehat{w}^*_{\widehat{\alpha}} - w^*_\alpha \widehat{w}_{\widehat{\alpha}}). \end{aligned} \tag{4.2}$$

Note the close resemblence of the first two lines in Ψ with the b and $\overline{b}$ ghost of (3.4) and (3.8), and that the last line of Ψ is gauge-invariant under (2.7) because of the constraints of (2.21).

Since Q is nilpotent, (4.1) is invariant under the BRST transformation of (2.14) and (2.27) and the resulting Noether charge is

$$Q = \int dz \eta_{\alpha\widehat{\alpha}} \lambda^{\alpha} J^{\widehat{\alpha}} + \int d\overline{z} \eta_{\alpha\widehat{\alpha}} \widehat{\lambda}^{\widehat{\alpha}} \overline{J}^{\alpha} \tag{4.3}$$

as before.

Using the identity of (3.7) and the BRST transformations of (2.14) and (2.27), it is straightforward to show that $Q\Psi$ is equal to the Lagrangian of (4.1). The BRST transformation of the first line of (4.2) is

$$\frac{1}{2}\left[\frac{\eta^{\alpha\widehat{\alpha}}(\gamma_a\widehat{\lambda})_{\widehat{\alpha}}(\gamma_b\lambda)_{\alpha}}{2(\eta\lambda\widehat{\lambda})}\overline{J}^a J^b + \eta_{\alpha\widehat{\alpha}}\overline{J}^{\alpha}J^{\widehat{\alpha}} - w_{\alpha}\overline{\nabla}\lambda^{\alpha} + \frac{1}{8(\eta\lambda\widehat{\lambda})}((w^*\gamma^{ab}\lambda)(\widehat{\lambda}\gamma_{ab}\overline{J}) + 2(w^*\lambda)(\widehat{\lambda}\overline{J}))\right], \tag{4.4}$$

the BRST transformation of the second line of (4.2) is

$$\frac{1}{2}\left[\frac{\eta^{\alpha\widehat{\alpha}}(\gamma_a\lambda)_{\alpha}(\gamma_b\widehat{\lambda})_{\widehat{\alpha}}}{2(\eta\lambda\widehat{\lambda})}J^a\overline{J}^b + \eta_{\alpha\widehat{\alpha}}\overline{J}^{\alpha}J^{\widehat{\alpha}} + \widehat{w}_{\widehat{\alpha}}\nabla\widehat{\lambda}^{\widehat{\alpha}} - \frac{1}{8(\eta\lambda\widehat{\lambda})}((\widehat{w}^*\gamma^{ab}\widehat{\lambda})(\lambda\gamma_{ab}J) + 2(\widehat{w}^*\widehat{\lambda})(\lambda J))\right], \tag{4.5}$$

and the BRST transformation of the third line of (4.2) is

$$\frac{1}{2}[2\eta^{\alpha\widehat{\alpha}}w^*_{\alpha}\widehat{w}^*_{\widehat{\alpha}} + w^*_{\alpha}\overline{J}^{\alpha} - \widehat{w}^*_{\widehat{\alpha}}J^{\widehat{\alpha}} - w_{\alpha}\overline{\nabla}\lambda^{\alpha} + \widehat{w}_{\widehat{\alpha}}\nabla\widehat{\lambda}^{\widehat{\alpha}} - 2\eta_{[ab][cd]}N^{ab}\widehat{N}^{cd}]. \tag{4.6}$$

It is interesting to note that the difference between the topological and $AdS_5 \times S^5$ actions of (4.1) and (2.11) is

$$S_{\rm top} - S_{AdS_5\times S^5} = \int d^2z\left[\frac{\eta^{\alpha\widehat{\alpha}}(\gamma_a\lambda)_{\alpha}(\gamma_b\widehat{\lambda})_{\widehat{\alpha}}}{4(\eta\lambda\widehat{\lambda})}(J^a\overline{J}^b - \overline{J}^a J^b) + \frac{1}{4}\eta_{\alpha\widehat{\beta}}(J^{\alpha}\overline{J}^{\widehat{\beta}} - \overline{J}^{\alpha}J^{\widehat{\beta}})\right], \tag{4.7}$$

where the pure spinors $(\lambda^{\alpha}, \widehat{\lambda}^{\widehat{\alpha}})$ choose a complex structure which allows the covariant construction of a Wess-Zumino term from the bosonic currents $(J^a, \overline{J}^a)$. Using $\lambda\gamma^a\lambda = \widehat{\lambda}\gamma^a\widehat{\lambda} = 0$ and the BRST transformation of (2.14), one can easily check that (4.7) is BRST-closed. And since (4.7) is antisymmetric in z and $\overline{z}$, it is clear that the stress tensor of $S_{\rm top}$ is equal to the $AdS_5 \times S^5$ stress tensor of (3.3).

One can formally define an analogous topological action in a flat Type II background as

$$\begin{aligned} S^{\rm flat}_{\rm top} &= \int d^2z\, Q(\Psi^{\rm flat}) \\ &= \int d^2z\left[\frac{\eta^{\alpha\widehat{\alpha}}(\gamma_a\lambda)_{\alpha}(\gamma_b\widehat{\lambda})_{\widehat{\alpha}}}{2(\eta\lambda\widehat{\lambda})}\Pi^a\overline{\Pi}^b - d_{\alpha}\overline{\partial}\theta^{\alpha} + \widehat{d}_{\widehat{\alpha}}\partial\widehat{\theta}^{\widehat{\alpha}} - w_{\alpha}\overline{\partial}\lambda^{\alpha} + \widehat{w}_{\widehat{\alpha}}\partial\widehat{\lambda}^{\widehat{\alpha}} + \eta^{\alpha\widehat{\alpha}}w^*_{\alpha}\widehat{w}^*_{\widehat{\alpha}}\right] \end{aligned} \tag{4.8}$$

where $\Pi^a = \partial x^a + \theta\gamma^a\partial\theta + \widehat{\theta}\gamma^a\partial\widehat{\theta}$, $\eta^{\alpha\widehat{\alpha}}$ is a constant bispinor, and

$$\begin{aligned} \Psi^{\rm flat} &= \frac{1}{2}(\eta\lambda\widehat{\lambda})^{-1}\widehat{\lambda}^{\widehat{\alpha}}\eta_{\alpha\widehat{\alpha}}\left(\frac{1}{2}\gamma_a^{\alpha\beta}\overline{\Pi}^a d_{\beta} + \frac{1}{4}(\gamma_{ab})^{\alpha}_{\beta}N^{ab}\overline{\partial}\theta^{\beta} + \frac{1}{4}J_{gh}\overline{\partial}\theta^{\alpha}\right) \\ &+\frac{1}{2}(\eta\lambda\widehat{\lambda})^{-1}\lambda^{\alpha}\eta_{\alpha\widehat{\alpha}}\left(-\frac{1}{2}\gamma_a^{\widehat{\alpha}\widehat{\beta}}\Pi^a\widehat{d}_{\widehat{\beta}} - \frac{1}{4}(\gamma_{ab})^{\widehat{\alpha}}_{\widehat{\beta}}\widehat{N}^{ab}\partial\widehat{\theta}^{\widehat{\beta}} - \frac{1}{4}\widehat{J}_{gh}\partial\widehat{\theta}^{\widehat{\alpha}}\right) \\ &+\frac{1}{2}\eta^{\alpha\widehat{\alpha}}(w_{\alpha}\widehat{w}^*_{\widehat{\alpha}} - w^*_{\alpha}\widehat{w}_{\widehat{\alpha}}). \end{aligned} \tag{4.9}$$

JHEP09(2009)051

The choice of $\eta^{\alpha\widehat{\alpha}}$ breaks Lorentz invariance for the Type IIB superstring, but for the Type IIA superstring, Lorentz invariance can be preserved by choosing $\eta^{\alpha\widehat{\alpha}} = \delta^{\alpha\widehat{\alpha}}$. Note that unlike the usual pure spinor action in a flat background, the topological action $S^{\text{flat}}_{\text{top}}$ is manifestly spacetime supersymmetric and satisfies

$$S^{\text{flat}}_{\text{top}} - S_{\text{flat}} = \int d^2z \left[\frac{\eta^{\alpha\widehat{\alpha}}(\gamma_a\lambda)_\alpha(\gamma_b\widehat{\lambda})_{\widehat{\alpha}}}{4(\eta\lambda\widehat{\lambda})}(\Pi^a\overline{\Pi}^b - \overline{\Pi}^a\Pi^b) - L_{\text{WZ}} \right] \tag{4.10}$$

where L_{WZ} is the standard Green-Schwarz Wess-Zumino term. However, unlike the topological $AdS_5 \times S^5$ action of (4.1), the topological action of (4.8) in a flat background is not well-defined since inverse powers of $(\eta\lambda\widehat{\lambda})$ are not allowed in the flat Hilbert space. As emphasized in section 3, the presence of inverse powers of $(\eta\lambda\widehat{\lambda})$ in a flat background would trivialize the BRST cohomology.

4.2 G/ G principal chiral model

In [9] and [10], an A-twisted $N = 2$ worldsheet supersymmetric sigma model constructed from the fermionic coset $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,2)\times\text{SO}(6)}$ was conjectured to describe the zero-radius limit of the $AdS_5 \times S^5$ superstring. This topological sigma model was related by a field redefinition to the $AdS_5 \times S^5$ sigma model of (2.11), but the BRST operators for the topological and $AdS_5 \times S^5$ sigma models were different. It was then shown in [11] that this $N = 2$ worldsheet supersymmetric sigma model constructed from the fermionic coset $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,2)\times\text{SO}(6)}$ could be obtained by gauge-fixing the G/G principal chiral model

$$S = Str \int d^2z(G^{-1}\partial G - A)(G^{-1}\overline{\partial}G - \overline{A}) = \int d^2z\ \eta_{\tilde{A}\tilde{B}}\ (J^{\tilde{A}} - A^{\tilde{A}})(\overline{J}^{\tilde{B}} - \overline{A}^{\tilde{B}}) \tag{4.11}$$

where G takes values in $\text{PSU}(2,2|4)$, $J = G^{-1}\partial G$ are the left-invariant currents, $\eta_{\tilde{A}\tilde{B}}$ is the $\text{PSU}(2,2|4)$ metric, and $(A, \overline{A})$ is a worldsheet gauge field taking values in the $\text{PSU}(2,2|4)$ Lie algebra. Although this G/G model appears to be trivial, it will be argued later that it contains non-trivial physical states because of boundary conditions on the non-compact $\text{PSU}(2,2|4)$ generators.

The action of (4.11) is invariant under the local $\text{PSU}(2,2|4)$ gauge transformations

$$\delta G = G\Omega, \quad \delta A = d\Omega + [A, \Omega], \tag{4.12}$$

and to obtain the supersymmetric sigma model based on the fermionic coset, one first uses the $\text{SO}(4,2) \times \text{SO}(6)$ generators of Ω to gauge away the bosonic elements in G so that G takes values in the fermionic coset $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,2)\times\text{SO}(6)}$. One then uses the fermionic generators of Ω to gauge-fix

$$A^{\alpha+} \equiv A^\alpha + iA^{\widehat{\alpha}} = 0, \quad \overline{A}^{\alpha-} \equiv \overline{A}^\alpha - i\overline{A}^{\widehat{\alpha}} = 0, \tag{4.13}$$

where $T_{\alpha+} \equiv T_\alpha + iT_{\widehat{\alpha}}$ are the 16 fermionic generators in the upper-right square of $\text{PSU}(2,2|4)$ and $T_{\alpha-} \equiv T_\alpha - iT_{\widehat{\alpha}}$ are the 16 fermionic generators in the lower-left square of $\text{PSU}(2,2|4)$.

JHEP09(2009)051

This fermionic gauge-fixing gives rise to bosonic ghosts $(Z^{\alpha -}, \overline{Z}^{\alpha +})$ and antighosts $(Y_{\alpha -}, \overline{Y}_{\alpha +})$ with the Faddeev-Popov action

$$S_{gh} = \int d^2 z[-Y_{\alpha -}\overline{\nabla} Z^{\alpha -} + \overline{Y}_{\alpha +}\nabla \overline{Z}^{\alpha +}] \tag{4.14}$$

and the BRST operator

$$Q = \int dz \eta_{\alpha\beta} Z^{\alpha -} J^{\beta +} + \int d\overline{z} \eta_{\alpha\beta} \overline{Z}^{\beta +} \overline{J}^{\alpha -} \tag{4.15}$$

where $\eta_{\alpha\beta} = (\gamma^{01234})_{\alpha\beta}$. Note that $Q^2 = 0$ without imposing pure spinor constraints on $Z^{\alpha -}$ and $\overline{Z}^{\alpha +}$ because $T_{\alpha +}$ and $T_{\alpha -}$ satisfy $\{T_{\alpha +}, T_{\beta +}\} = \{T_{\alpha -}, T_{\beta -}\} = 0$. In this gauge, the action of (4.11) reduces to an A-twisted $N = 2$ worldsheet supersymmetric sigma model where $(Z^{\alpha -}, \overline{Z}^{\alpha +}, Y_{\alpha -}, \overline{Y}_{\alpha +})$ are the bosonic worldsheet superpartners to the fermionic coset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,2)\times \mathrm{SO}(6)}$ and (4.15) is the scalar worldsheet supersymmetry generator.

Although the BRST operator of (4.15) in this gauge-fixing is different from the original $AdS_5 \times S^5$ BRST operator of (2.13), it will now be shown that there is an alternative gauge-fixing of the G/G model of (4.11) which leads to the topological action of (4.1) and which has the same BRST operator as (2.13). To obtain the topological action of (4.1) from (4.11), one first uses the local $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ gauge invariances of (4.12) to gauge-fix G to take values in the Metsaev-Tseytlin coset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times \mathrm{SO}(5)}$. One next uses the fermionic gauge transformations of (4.12) to gauge-fix

$$A^{\widehat{\alpha}} = 0, \quad \overline{A}^{\alpha} = 0, \tag{4.16}$$

which gives rise to unconstrained bosonic ghosts $(Z^{\alpha}, \overline{Z}^{\widehat{\alpha}})$ and antighosts $(Y_{\alpha}, \overline{Y}_{\widehat{\alpha}})$ with the Faddeev-Popov action

$$S_{gh} = \int d^2 z[-Y_{\alpha}\overline{\nabla} Z^{\alpha} + \overline{Y}_{\widehat{\alpha}}\nabla \overline{Z}^{\widehat{\alpha}}] \tag{4.17}$$

where $\overline{\nabla} Z^{\alpha} = \overline{\partial} Z^{\alpha} + \frac{1}{2}\overline{A}^{[ab]}(\gamma_{[ab]} Z)^{\alpha}$ and $\nabla \overline{Z}^{\widehat{\alpha}} = \partial \overline{Z}^{\widehat{\alpha}} + \frac{1}{2} A^{[ab]}(\gamma_{[ab]}\overline{Z})^{\widehat{\alpha}}$. Since $\{T_{\alpha}, T_{\beta}\}$ and $\{T_{\widehat{\alpha}}, T_{\widehat{\beta}}\}$ are nonzero and Z^{α} and $\overline{Z}^{\widehat{\alpha}}$ are unconstrained, the BRST operator

$$Q = \int dz \eta_{\alpha\widehat{\alpha}} Z^{\alpha} J^{\widehat{\alpha}} + \int d\overline{z} \eta_{\alpha\widehat{\alpha}} \overline{Z}^{\widehat{\alpha}} \overline{J}^{\alpha} \tag{4.18}$$

implied by this gauge-fixing would not be nilpotent.

However, one still has ten bosonic gauge transformations of (4.12) which need to be gauge-fixed. Although one could naively use these gauge transformations to gauge away the remaining bosonic components of G, this will be argued later to be inconsistent with the boundary conditions of the $\mathrm{PSU}(2,2|4)$ gauge parameters. Instead, one can use these ten gauge transformations to gauge-fix 5 components of A^a and 5 components of $\overline{A}^a$ to zero. The choice of which five components of A^a and $\overline{A}^a$ are gauge-fixed will be correlated with the bosonic ghosts $(Z^{\alpha}, \overline{Z}^{\widehat{\alpha}})$ in such a manner that the resulting BRST operator is nilpotent. Using an $AdS_5 \times S^5$ adaptation of the "extended pure spinor formalism" of Aisaka and Kazama [14], this BRST operator will then be shown to have the same cohomology as the original $AdS_5 \times S^5$ BRST operator of (2.13).

To determine which components of A^a should be gauge-fixed, note that $(\gamma_a)_{\alpha\beta} Z^\alpha Z^\beta$ is a null vector which decomposes under SO(4,1) $\times$ SO(5) into

$$\Phi_I = (\gamma_I)_{\alpha\beta} Z^\alpha Z^\beta, \quad \Psi_{\tilde{I}} = (\gamma_{\tilde{I}})_{\alpha\beta} Z^\alpha Z^\beta \tag{4.19}$$

for $I = 0$ to 4 and $\tilde{I} = 5$ to 9. Furthermore, if Φ_I is zero for $I = 0$ to 4, then $\Psi_{\tilde{I}}$ is also zero for $\tilde{I} = 5$ to 9. This can be seen from the fact that a pure spinor contains 11 independent components and therefore satisfies 5 independent constraints. So if $\Phi_I = 0$ for $I = 0$ to 4, Z^α will be a pure spinor, which implies that $\Psi_{\tilde{I}} = 0$ for $\tilde{I} = 5$ to 9. Since $\Phi_I = 0$ implies $\Psi_{\tilde{I}} = 0$, there exists an invertible matrix $M_{\tilde{I}}^J(Z)$ such that

$$\Psi_{\tilde{I}}(Z) = M_{\tilde{I}}^J(Z)\ \Phi_J(Z). \tag{4.20}$$

It will be convenient to define the matrix $\mathcal{N}_a^I(Z)$ such that

$$\gamma_{a\alpha\beta} Z^\alpha Z^\beta = \mathcal{N}_a^I(Z) \Phi_I(Z) \tag{4.21}$$

where $\mathcal{N}_a^I = \delta_a^I$ for $a = 0$ to 4, and $\mathcal{N}_a^I = M_a^I$ for $a = 5$ to 9. Since $\eta^{ab}(Z\gamma_a Z)(Z\gamma_b Z) = 0$ and since the Φ_I's are independent, $\mathcal{N}_a^I$ satisfies the identity

$$\eta^{ab} \mathcal{N}_a^I \mathcal{N}_b^J = 0. \tag{4.22}$$

Similarly, one can define the matrix $\overline{\mathcal{N}}_a^I(\overline{Z})$ such that

$$\gamma_{a\widehat{\alpha}\widehat{\beta}} \overline{Z}^{\widehat{\alpha}} \overline{Z}^{\widehat{\beta}} = \overline{\mathcal{N}}_a^I(\overline{Z}) \overline{\Phi}_I(\overline{Z}), \quad \eta^{ab} \overline{\mathcal{N}}_a^I \overline{\mathcal{N}}_b^J = 0. \tag{4.23}$$

One now uses $\mathcal{N}_a^I(Z)$ and $\overline{\mathcal{N}}_a^I(\overline{Z})$ to choose the gauge-fixing conditions

$$\mathcal{N}_a^I(Z) A^a = 0, \quad \overline{\mathcal{N}}_a^I(\overline{Z}) \overline{A}^a = 0 \tag{4.24}$$

for $I = 0$ to 4. With this gauge-fixing, the G/G model of (4.11) becomes

$$\begin{aligned} S = \int d^2 z \Big[& \eta_{\tilde{A}\tilde{B}} (J^{\tilde{A}} - A^{\tilde{A}})(\overline{J}^{\tilde{B}} - \overline{A}^{\tilde{B}}) + \overline{f}_I \mathcal{N}_a^I A^a + f_I \overline{\mathcal{N}}_a^I \overline{A}^a + f_\alpha \overline{A}^\alpha + \overline{f}_{\widehat{\alpha}} A^{\widehat{\alpha}} \\ & - Y_\alpha (\overline{\nabla} Z^\alpha - \eta^{\alpha\widehat{\alpha}} (\overline{Z}\gamma_a)_{\widehat{\alpha}} \overline{A}^a - c^a \gamma_a^{\alpha\beta} \eta_{\beta\widehat{\beta}} \overline{A}^{\widehat{\beta}}) + \overline{Y}_{\widehat{\alpha}} (\nabla \overline{Z}^{\widehat{\alpha}} + \eta^{\alpha\widehat{\alpha}} (Z\gamma_a)_\alpha A^a + c^a \gamma_a^{\widehat{\alpha}\widehat{\beta}} \eta_{\beta\widehat{\beta}} A^\beta) \\ & - b_I \overline{\mathcal{N}}_a^I (\overline{\nabla} c^a + (\overline{Z}\gamma^a)_{\widehat{\alpha}} \overline{A}^{\widehat{\alpha}} + (Z\gamma^a)_\alpha \overline{A}^\alpha) - \overline{b}_I \mathcal{N}_a^I (\nabla c^a + (Z\gamma^a)_\alpha A^\alpha + (\overline{Z}\gamma^a)_{\widehat{\alpha}} A^{\widehat{\alpha}}) \Big] \end{aligned} \tag{4.25}$$

and the BRST operator is

$$\begin{aligned} Q = & \int dz [Z^\alpha f_\alpha + b_I R^{IJ} \Phi_J + c^a (\overline{\mathcal{N}}_a^I f_I + K_a)] \\ & + \int d\overline{z} [\overline{Z}^{\widehat{\alpha}} \overline{f}_{\widehat{\alpha}} + \overline{b}_J R^{IJ} \overline{\Phi}_I + c^a (\mathcal{N}_a^I \overline{f}_I + \overline{K}_a)] \end{aligned} \tag{4.26}$$

where $(f_I, \overline{f}_I, f_\alpha, \overline{f}_{\widehat{\alpha}})$ are Lagrange multipliers which impose the gauge-fixing conditions, $(c^a, Z^\alpha, \overline{Z}^{\widehat{\alpha}})$ and $(b_I, \overline{b}_I, Y_\alpha, \overline{Y}_{\widehat{\alpha}})$ are the Faddeev-Popov ghosts and antighosts coming from the gauge-fixing of (4.16) and (4.24), and

$$R^{IJ} \equiv \eta^{ab}\overline{\mathcal{N}}^I_a \mathcal{N}^J_b, \quad K_a \equiv \eta_{\alpha\widehat{\alpha}}(\gamma_a Y)^\alpha \overline{Z}^{\widehat{\alpha}}, \quad \overline{K}_a \equiv \eta_{\alpha\widehat{\alpha}}(\gamma_a \overline{Y})^{\widehat{\alpha}} Z^\alpha. \tag{4.27}$$

After integrating out the worldsheet gauge fields and Lagrange multipliers which satisfy auxiliary equations of motion, (4.25) reduces to

$$\begin{aligned} S = \int d^2z \Big[& J^a \mathcal{N}^I_a R^{-1}_{IJ} \overline{\mathcal{N}}^J_b \overline{J}^b + \eta_{\alpha\widehat{\alpha}} \overline{J}^\alpha J^{\widehat{\alpha}} - Y_\alpha(\overline{\nabla} Z^\alpha + \cdots) + \overline{Y}_{\widehat{\alpha}}(\nabla \overline{Z}^{\widehat{\alpha}} + \cdots) \\ & - b_I \overline{\mathcal{N}}^I_a(\overline{\nabla} c^a + \cdots) + \overline{b}_I \mathcal{N}^I_a(\nabla c^a + \cdots) - \eta^{[ab][cd]}\Big(\frac{1}{2} Y\gamma_{ab}Z + b_I \overline{\mathcal{N}}^I_a c_b\Big)\Big(\frac{1}{2}\overline{Y}\gamma_{cd}\overline{Z} + \overline{b}_J \mathcal{N}^J_c c_d\Big)\Big] \end{aligned} \tag{4.28}$$

with the BRST operator

$$\begin{aligned} Q = & \int dz [\eta_{\alpha\widehat{\alpha}} Z^\alpha J^{\widehat{\alpha}} + b_I R^{IJ}\Phi_J + c^a \overline{\mathcal{N}}^I_a R^{-1}_{JI} \mathcal{N}^J_b (J^b - K^b) + c^a K_a] \\ & + \int d\overline{z} [\eta_{\alpha\widehat{\alpha}} \overline{Z}^{\widehat{\alpha}} \overline{J}^\alpha + \overline{b}_I R^{JI} \overline{\Phi}_J + c^a \mathcal{N}^I_a R^{-1}_{IJ} \overline{\mathcal{N}}^J_b (\overline{J}^b - \overline{K}^b) + c^a \overline{K}_a] \end{aligned} \tag{4.29}$$

where

$$\begin{aligned} \overline{\nabla} Z^\alpha &= \overline{\partial} Z^\alpha + \frac{1}{2}\overline{J}^{[ab]}(\gamma_{ab}Z)^\alpha, & \nabla \overline{Z}^{\widehat{\alpha}} &= \partial \overline{Z}^{\widehat{\alpha}} + \frac{1}{2} J^{[ab]}(\gamma_{ab}\overline{Z})^{\widehat{\alpha}}, \\ \overline{\nabla} c^a &= \overline{\partial} c^a + \overline{J}^{[ab]} c_b, & \nabla c^a &= \partial c^a + J^{[ab]} c_b, \end{aligned} \tag{4.30}$$

and R^{-1}_{IJ} is the inverse matrix to $R^{IJ} \equiv \eta^{ab}\overline{\mathcal{N}}^I_a \mathcal{N}^J_b$ satisfying $R^{-1}_{IJ} R^{JK} = \delta^K_I$. Note that the last term of (4.28) comes from integrating out $A^{[ab]}$ and $\overline{A}^{[ab]}$ which converts the covariant derivatives in (4.17) into the covariant derivatives of (4.30).

As shown in [14] using "homological perturbation" theory, the BRST operator of (4.29) is equivalent to the BRST operator $Q = \int dz \eta_{\alpha\widehat{\alpha}} \lambda^\alpha J^{\widehat{\alpha}} + \int d\overline{z} \eta_{\alpha\widehat{\alpha}} \widehat{\lambda}^{\widehat{\alpha}} \overline{J}^\alpha$ where the terms $\int dz b_I R^{IJ}\Phi_J$ and $\int d\overline{z}\overline{b}_I R^{JI}\overline{\Phi}_J$ in (4.29) have been used to strongly impose the constraints $\Phi_I = \overline{\Phi}_I = 0$ and to gauge $c^a = 0$. In the presence of the constraints $\Phi_I = \overline{\Phi}_I = 0$, the ghosts Z^α and $\overline{Z}^{\widehat{\alpha}}$ reduce to pure spinors which will be called λ^α and $\widehat{\lambda}^{\widehat{\alpha}}$. Furthermore, $\Phi_I = \overline{\Phi}_I = 0$ implies that $(\lambda\gamma^a)_\alpha \mathcal{N}^I_a = (\widehat{\lambda}\gamma^a)_{\widehat{\alpha}} \overline{\mathcal{N}}^I_a = 0$, and that

$$\mathcal{N}^I_a R^{-1}_{IJ} \overline{\mathcal{N}}^J_b = \frac{(\lambda\gamma_a)_\alpha \eta^{\alpha\widehat{\alpha}} (\widehat{\lambda}\gamma_b)_{\widehat{\alpha}}}{2(\eta\lambda\widehat{\lambda})} \tag{4.31}$$

where the normalization of (4.31) is fixed by $\eta^{ab}(\mathcal{N}^I_a R^{-1}_{IJ} \overline{\mathcal{N}}^J_b) = R^{-1}_{IJ} R^{JI} = 5$. Finally, when $c^a = 0$ and $\Phi_I = \overline{\Phi}_I = 0$, it is straightforward to check that the ... terms in (4.28) are zero and that (4.28) coincides with (4.1).

So it has been shown that the topological $AdS_5 \times S^5$ action of (4.1) and BRST operator of (4.3) can be obtained from the G/G principal chiral model of (4.11) by choosing the gauge

$$A^{\widehat{\alpha}} = \overline{A}^\alpha = \mathcal{N}^I_a(Z) A^a = \overline{\mathcal{N}}^I_a(\overline{Z})\overline{A}^a = 0, \tag{4.32}$$

JHEP09(2009)051

where the tensors $\mathcal{N}^I_a(Z)$ and $\overline{\mathcal{N}}^I_a(\overline{Z})$ are constructed from the bosonic Faddeev-Popov ghosts. In the next subsection, it will be argued that this topological model describes the zero-radius limit of the $AdS_5 \times S^5$ superstring which is dual to free $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills theory.

4.3 Physical states

If the topological model of (4.1) is to describe the zero radius limit of the $AdS_5 \times S^5$ superstring, physical states in the BRST cohomology of this model should correspond to gauge-invariant super-Yang-Mills operators at zero 't Hooft coupling. Naively, the G/G model has no physical states since one could use the local PSU(2, 2|4) gauge invariance of (4.12) to gauge $G = 1$. In this gauge, there are no propagating ghosts and the equations of motion for the worldsheet gauge field are simply $A^{\tilde{A}} = \overline{A}^{\tilde{A}} = 0$.

However, because of the non-compact generators in PSU(2, 2|4), there are subtleties in choosing the gauge $G = 1$. Suppose one parameterizes the PSU(2, 2|4) matrix G as

$$G = \exp(x^m P_m + \theta^j_\mu q^\mu_j + \overline{\theta}^{\dot\mu}_j \overline{q}^j_{\dot\mu}) \exp(-yD + \phi_{jk} R^{jk} + t_{mn} M^{mn}) \exp(h_m K^m + \xi^\mu_j s^j_\mu + \overline{\xi}^j_{\dot\mu} \overline{s}^{\dot\mu}_j) \tag{4.33}$$

where $(P_m, q^\mu_j, \overline{q}^j_{\dot\mu})$ are the $\mathcal{N} = 4$ $d = 4$ translation and supersymmetry generators for $m =$ 0 to 3, $j = 1$ to 4 and $(\mu, \dot\mu) = 1$ to 2, (D, R^{jk}, M^{mn}) are the dilatation, SO(6) R-symmetry, and SO(3, 1) Lorentz generators, and $(K^m, s^j_\mu, \overline{s}^{\dot\mu}_j)$ are the conformal and superconformal generators. With this parameterization of G, the global PSU(2, 2|4) isometries $\delta G = \Sigma G$ transform the variables $(x^m, \theta^j_\mu, \overline{\theta}^{\dot\mu}_j)$ into themselves in the standard $\mathcal{N} = 4$ $d = 4$ superconformal manner. Furthermore, using the relations

$$K^m e^{-yD} = e^{-yD}(e^{-y} K_m), \quad s^j_\mu e^{-yD} = e^{-yD}(e^{-\frac{1}{2}y} s^j_\mu), \quad \overline{s}^{\dot\mu}_j e^{-yD} = e^{-yD}(e^{-\frac{1}{2}y} \overline{s}^{\dot\mu}_j), \tag{4.34}$$

one finds that in the limit $y \to \infty$, the variables $(h^m, \xi^\mu_j, \overline{\xi}^j_{\dot\mu})$ are invariant under the global PSU(2, 2|4) transformations. So it is natural to identify $(x^m, \theta^j_\mu, \overline{\theta}^{\dot\mu}_j)$ as parameterizing the boundary of $AdS_5 \times S^5$ in the limit where $y \to \infty$.

Under the local PSU(2, 2|4) gauge transformations $\delta G = G\Omega$ of (4.12), one could naively gauge-fix to zero all the variables in (4.33). However, using the relations

$$e^{-yD} P_m = (e^{-y} P_m) e^{-yD}, \quad e^{-yD} q^\mu_j = (e^{-\frac{1}{2}y} q^\mu_j) e^{-yD}, \quad e^{-yD} \overline{q}^j_{\dot\mu} = (e^{-\frac{1}{2}y} \overline{q}^j_{\dot\mu}) e^{-yD}, \tag{4.35}$$

one finds that in the limit where $y \to \infty$, the variables $(x^m, \theta^j_\mu, \overline{\theta}^{\dot\mu}_j)$ are invariant under these gauge transformations. So assuming that the gauge parameters in Ω of (4.12) do not blow up when $y \to \infty$, the boundary of $AdS_5 \times S^5$ is gauge-invariant and cannot be gauged away. The G/G principal chiral model could therefore have physical states which depend non-trivially on the $AdS_5 \times S^5$ boundary variables $(x^m, \theta^j_\mu, \overline{\theta}^{\dot\mu}_j)$ when $y \to \infty$.[2]

[2] Using the gauge-fixing to the fermionic coset, the x^m variables were gauged to zero which explains why it was difficult to construct physical vertex operators in terms of the fermionic coset variables. In [11], it was conjectured that the non-trivial physical states could emerge after including a kinetic term for the worldsheet gauge field. However, this conjecture appears to be incorrect since the kinetic term goes to zero in the infrared limit of the sigma model. I would like to thank A. Polyakov for correcting this point and for suggesting that the topological action should be perturbed by an appropriate radius-dependent operator.

In fact, it is easy to verify that in the gauge of (4.32) where G takes values in the Metsaev-Tseytlin coset $g \in \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times \mathrm{SO}(5)}$, there are such physical states in the BRST cohomology. Using the topological action of (4.1), the BRST operator of (4.3) transforms

$$Qg = g(\lambda^\alpha T_\alpha + \widehat{\lambda}^{\widehat{\alpha}} T_{\widehat{\alpha}}) \tag{4.36}$$

in precisely the same manner as in the $AdS_5 \times S^5$ formalism of section 2. So the supergravity vertex operator $V = \lambda^\alpha \widehat{\lambda}^{\widehat{\alpha}} A_{\alpha\widehat{\alpha}}(x, \theta, \widehat{\theta})$ is in the BRST cohomology of the topological model when $A_{\alpha\widehat{\alpha}}$ satisfies the equations of motion and gauge invariances of (2.31) and (2.33).

These supergravity vertex operators depend only on the zero modes of the worldsheet variables and correspond to the half-BPS Yang-Mills operators. Vertex operators corresponding to non-BPS Yang-Mills operators are expected to depend on non-zero modes of the worldsheet variables and will be more difficult to explicitly construct. Nevertheless, it will be conjectureed that these non-BPS vertex operators can be obtained from BPS vertex operators by transforming the worldsheet variables described by the Metsaev-Tseytlin coset $g \in \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times \mathrm{SO}(5)}$ as

$$\delta g(\sigma) = \Sigma(\sigma) g(\sigma) \tag{4.37}$$

where $0 \leq \sigma < 2\pi$ is the closed string parameter and $\Sigma(\sigma)$ is a $\mathrm{PSU}(2,2|4)$ transformation which is allowed to depend on σ. Although an explicit construction of these non-BPS vertex operators is unknown, they should be defined such that they are ghost-number 2 elements in the BRST cohomology as usual in the pure spinor formalism.

Since (4.37) acts by left multiplication and the BRST transformation of (4.36) acts by right multiplication, BRST transformations commute with (4.37). So $QV(g) = 0$ implies that $QV(g + \delta g) = 0$ where δg is defined in (4.37). When Σ is independent of σ, (4.37) is a global $\mathrm{PSU}(2,2|4)$ transformation which takes half-BPS vertex operators into half-BPS vertex operators. But when Σ depends on σ, (4.37) can take half-BPS vertex operators into non-BPS vertex operators which depend on non-zero modes of the worldsheet variables. Although (4.37) does not leave invariant the topological action of (4.1) when $\partial_\sigma \Sigma$ is nonzero, the change of the topological action is BRST-trivial and can be expressed as $\delta S = \int d^2 z Q[\Psi(g + \delta g) - \Psi(g)]$ where Ψ is defined in (4.9). So the transformation of (4.37) takes physical states into physical states.

To see an example where (4.37) transforms a physical half-BPS vertex operator into a physical non-BPS vertex operator, consider the half-BPS vertex operator $|0\rangle_J$ corresponding to the long gauge-invariant super-Yang-Mills operator

$$Tr(Z^J) \tag{4.38}$$

with large R-charge J where Z is the scalar at $x^m = 0$ with R-charge $+1$ with respect to a $\mathrm{U}(1)$ direction of $\mathrm{SO}(6)$. To be explicit, choose $Z = \phi_{12}$ where ϕ_{jk} are the six Yang-Mills scalars and J is the charge with respect to the $\mathrm{U}(1)$ generator $\frac{1}{2}(R_1^1 + R_2^2 - R_3^3 - R_4^4)$. The operator of (4.38) is invariant under all $\mathrm{PSU}(2,2|4)$ transformations of (4.37) except for the four translations P_m, the four R-symmetry generators $(R_3^1, R_3^2, R_4^1, R_4^2)$, and the eight supersymmetry generators $(q_3^\mu, q_4^\mu, \overline{q}_{\dot{\mu}}^1, \overline{q}_{\dot{\mu}}^2)$. Under these eight bosonic and eight fermionic transformations, the operator of (4.38) transforms in the same manner as in a Ramond-Ramond plane-wave background when acted on with the eight bosonic and eight fermionic light-cone oscillators [15].

To be more explicit, suppose that $(\Sigma_n)^k_j$ transforms $g(\sigma)$ as $\delta g(\sigma) = e^{in\sigma} R^k_j g(\sigma)$. Then $(\Sigma_n)^1_3 |0\rangle_J$ is the vertex operator corresponding to the long gauge-invariant Yang-Mills operator

$$\sum_{m=1}^{J} Tr(Z^m \,\phi_{32} Z^{J-m}) e^{2\pi i n \frac{m}{J}}. \tag{4.39}$$

As in a plane-wave background, this operator vanishes by cyclicity of the trace so one needs at least two σ-dependent transformations to construct a physical states which satisfies $L_0 - \overline{L}_0 = 0$. For example, $(\Sigma_{-n})^1_4 (\Sigma_n)^1_3 |0\rangle_J$ is the non-BPS vertex operator corresponding to the long gauge-invariant Yang-Mills operator

$$\sum_{m=1}^{J} Tr(\phi_{42} Z^m \,\phi_{32} Z^{J-m}) e^{2\pi i n \frac{m}{J}}. \tag{4.40}$$

The spectrum of these non-BPS operators is easily computed using the PSU(2, 2|4) algebra. For example, $[D - J, R^1_3] = R^1_3$ and $[D - J, R^1_4] = R^1_4$ where D is the dilatation generator. So the state $(\Sigma_{-n})^1_4 (\Sigma_n)^1_3 |0\rangle_J$ has eigenvalue $D - J = 2$ which is independent of n. This agrees with the expected result at zero 't Hooft coupling since the large R-charge formula for the eigenvalue of the n^{th} oscillator mode is

$$(D - J)_n = \sqrt{1 + \frac{4\pi g_s N}{J} n^2} \tag{4.41}$$

which is independent of n when $g_s N = 0$.

4.4 Scattering amplitudes and open-closed duality

If the topological action S_{top} of (4.1) describes the zero-radius limit of the $AdS_5 \times S^5$ superstring, the $AdS_5 \times S^5$ superstring at infinitesimal radius r should be described by the action

$$S_r = S_{\text{top}} + r^2 S_{\text{AdS}} \tag{4.42}$$

where S_{AdS} is the vertex operator for the radius modulus and is also the original $AdS_5 \times S^5$ action of (2.11). Since S_{top} and S_{AdS} are both invariant under the BRST transformation generated by (2.13), (4.42) is also BRST invariant.[3] Note that one could also consider the action $S_r = t S_{\text{top}} + r^2 S_{\text{AdS}}$ where t is a constant, but since S_{top} is BRST-trivial, the theory must be independent of the value of t.

The Maldacena conjecture predicts that perturbative superstring scattering amplitudes computed in the background of (4.42) should coincide with perturbative correlation functions of gauge-invariant super-Yang-Mills operators at small 't Hooft coupling. Although it is not yet known how to compute topological string amplitudes in the background of (4.42),

[3] Using the previous proposal of S_{top} based on the fermionic coset, such a perturbation of S_{top} would not be allowed since the topological and $AdS_5 \times S^5$ BRST operators were different.

JHEP09(2009)051

a handwaving argument will be sketched based on open-closed topological duality that such amplitudes should agree with the analogous super-Yang-Mills computations. If this handwaving argument could be made rigorous, it would provide a proof of the Maldacena conjecture at small 't Hooft coupling.

The handwaving argument is closely related to ideas in [16] and [17, 18] which describe open-closed topological duality in the context of the Kontsevitch model and Chern-Simons theory. The action S_{top} of (4.1) describes a closed topological string theory, and one can define an open topological string theory by placing M $D3$ branes at the boundary of AdS_5. As usual, the D_3 brane boundary conditions are Dirichlet for the $(x^4, \ldots, x^9)$ variables, Neumann for the $(x^0, \ldots, x^3)$ variables, and

$$\widehat{\lambda}^{\widehat{\alpha}} = (\gamma_{0123})^{\widehat{\alpha}}_{\alpha} \lambda^{\alpha}, \quad \widehat{w}_{\widehat{\alpha}} = (\gamma_{0123})^{\alpha}_{\widehat{\alpha}} w_{\alpha}, \tag{4.43}$$

for the pure spinor variables. Furthermore, the fermionic boundary conditions imply that $J^{\widehat{\alpha}} = (\gamma_{0123})^{\widehat{\alpha}}_{\alpha} \overline{J}^{\alpha}$, so the BRST operator satisfies $Q_L = Q_R$ on the boundary.

As discussed at the end of subsection (3.3), (4.43) implies that $(\eta\lambda\widehat{\lambda}) = \lambda\gamma^4\lambda = 0$, so one needs to introduce non-minimal variables on the boundary. These non-minimal variables turn the zero mode measure factor into the same measure factor as in a flat background which is the $d = 4$ dimensional reduction of

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1. \tag{4.44}$$

One might be worried that the term

$$\frac{(\lambda\gamma_a)_{\alpha}\eta^{\alpha\widehat{\alpha}}(\widehat{\lambda}\gamma_b)_{\widehat{\alpha}}}{2(\eta\lambda\widehat{\lambda})} J^a \overline{J}^b \tag{4.45}$$

in the action of (4.1) becomes singular on the boundary where $(\eta\lambda\widehat{\lambda}) = 0$, but the numerator $(\lambda\gamma_a)_{\alpha}\eta^{\alpha\widehat{\alpha}}(\widehat{\lambda}\gamma_b)_{\widehat{\alpha}}$ also vanishes on the boundary where it is proportional to $\lambda\gamma^a\gamma^4\gamma^b\lambda = 0$.

The first step in the open-closed duality argument is that the only physical open string states on the M D_3 branes are massless $\mathrm{U}(M)$ $\mathcal{N} = 4$ super-Yang-Mills states. It is clear that these super-Yang-Mills states are in the spectrum since the vertex operator $V = \lambda^{\alpha} A_{\alpha}(x, \theta)$ is in the open string BRST cohomology when $A_{\alpha}(x, \theta)$ satisfies the $d = 4$ dimensional reduction of the $d = 10$ linearized super-Yang-Mills equations of motion. However, the absence of other states in the open string BRST cohomology remains to be proven. Nevertheless, it is reasonable that there are no other physical open string states since the D_3 branes on the AdS_5 boundary preserve $\mathrm{PSU}(2, 2|4)$ invariance, so any other such states would have to preserve $\mathcal{N} = 4$ $d = 4$ superconformal invariance and transform in the adjoint representation of $\mathrm{U}(M)$.

The next step in the argument is that the open string field theory action given by

$$\mathcal{S} = \frac{1}{g^2}\langle VQV + \frac{2}{3}VVV\rangle \tag{4.46}$$

reproduces the $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills field theory action where V is the off-shell open string field, g is the square-root of the closed string coupling constant g_s, and the

JHEP09(2009)051

zero-mode measure factor in (4.46) is the $d = 4$ dimensional reduction of (4.44). This step is reasonable since, as in the Chern-Simons topological string [33], one expects the Feynman diagrams of the open topological string to reduce to the Feynman diagrams of the massless field theory. And as shown in [30, 31], the $d = 10$ super-Yang-Mills field theory action (or its dimensional reduction) can be expressed as $\mathcal{S} = \frac{1}{g^2}\langle VQV + \frac{2}{3}VVV\rangle$ where $V = \lambda^\alpha A_\alpha(x,\theta)$, $A_\alpha(x,\theta)$ is an off-shell $d = 10$ spinor superfield, $Q = \lambda^\alpha D_\alpha$, D_α is the $d = 10$ supersymmetric derivative, and $\langle\ \ \rangle$ is the zero mode measure factor of (4.44). Furthermore, it will be assumed that as in the Chern-Simons topological string [33], closed string states decouple from open string states and do not contribute to open topological string scattering amplitudes.

So when $r = 0$ in (4.42), it has been argued that the open string field theory for M D_3 branes at the boundary describes $\mathrm{U}(M)$ super-Yang-Mills theory with coupling constant $g = \sqrt{g_s}$. The final step in the argument is that adding the $r^2 S_{\mathrm{AdS}}$ perturbation to S_{top} in (4.42) affects the open string field theory by shifting the 't Hooft coupling constant. This step has an analog in the open-closed duality of [16] where parameters of the closed string background of topological gravity were shown to affect the open string field theory by shifting parameters in the Kontsevitch matrix model.

The justification for this step is that insertion of a closed string vertex operator at a puncture in an open topological string amplitude can be replaced by expanding the puncture into a hole and inserting an appropriate D-brane boundary state [16, 34]. For an arbitrary closed string vertex operator, the corresponding D-brane boundary state may be difficult to construct. But for the closed string vertex operator S_{AdS} which is $\mathrm{PSU}(2,2|4)$ invariant, it seems reasonable to assume that the corresponding D-brane boundary state is proportional to a D_3 brane at the AdS_5 boundary. Note that the proportionality constant $f(r)$ must go to zero when $r \to 0$ in order to be consistent with the assumed decoupling of closed string states from open string states in the topological string. So inserting the closed string vertex operator S_{AdS} at a puncture in an open topological string amplitude should be equivalent to expanding the puncture to a D_3 brane hole and multiplying by a factor of $f(r)$.

Perturbing the background from $S_{\mathrm{top}} \to S_{\mathrm{top}} + r^2 S_{\mathrm{AdS}}$ is equivalent to inserting an exponential set of closed string vertex operators, and for each open string diagram with h holes and p punctures, the scattering amplitude is proportional to

$$(g^2 M)^h (r^2)^p \tag{4.47}$$

where $(g^2M)^h$ comes from the usual $(\lambda_{'tHooft})^h$ factor in the 't Hooft expansion. Replacing the punctures by D-brane holes and including the proportionality constant of $f(r)$, the open string scattering amplitude with H holes is proportional to

$$\sum_{h+p=H} \frac{(h+p)!}{h!p!}(g^2M)^h (r^2 f(r))^p = (g^2 M + r^2 f(r))^H \tag{4.48}$$

where the factor of $\frac{(h+p)!}{h!p!}$ comes from the different ways to split the H holes into h holes and p punctures.

So in the background of (4.42), it has been argued that the open string field theory for M D_3 branes on the AdS_5 boundary describes super-Yang-Mills theory where the 't

JHEP09(2009)051

Hooft coupling is shifted from g^2M to $g^2M + r^2 f(r)$. Note that if one could show that $f(r)$ were equal to r^2, this argument would imply that the 't Hooft coupling is equal to r^4 when $M = 0$. So the relation $\lambda_{'tHooft} = r^4$ would be valid both at small and large radius.

5 Conclusions and discussion

In the first half of this paper, it was shown that $(\eta\lambda\widehat{\lambda})$ is in the BRST cohomology in an $AdS_5 \times S^5$ background, which implies that the left and right-moving pure spinor ghosts can be treated as complex conjugate variables. This eliminates the need for non-minimal variables and simplifies the zero-mode measure factor and b ghost.

In the second half of this paper, a BRST-trivial version of the $AdS_5 \times S^5$ action was constructed by gauge-fixing a G/G principal chiral model where $G = \mathrm{PSU}(2,2|4)$. This topological action was argued to describe the zero radius limit which is dual to free super-Yang-Mills, and perturbing the topological action by the vertex operator for the radius modulus was conjectured to describe super-Yang-Mills at small 't Hooft coupling.

One possible method for proving this conjecture uses open-closed topological string duality along the lines proposed in the previous subsection. However, a more direct method would be to compute the topological closed string amplitudes and compare with the perturbative Feynman diagrams of the super-Yang-Mills field theory. In [11], a connection was found between networks of Wilson lines constructed from worldsheet gauge fields in the G/G model and the propagators and vertices of $\mathcal{N} = 4$ super-Yang-Mills Feynman diagrams. It would be very exciting if amplitude computations in the topological model could be related to counting these Wilson line networks in the G/G model.

Although it is well-understood how to compute scattering amplitudes with conventional topological string theories, the topological model of (4.1) has some new features which have not yet been studied. Unlike the usual topological strings where the complex structure of the target spacetime is fixed, the complex structure of the target spacetime in (4.1) is determined dynamically by the pure spinors $(\lambda^\alpha, \widehat{\lambda}^{\widehat{\alpha}})$ which choose a U(5) subgroup of the (Wick-rotated) SO(10) Lorentz group. This can be seen from the kinetic term for the x's in the topological action which, to quadratic order, is $\int d^2z (2\eta\lambda\widehat{\lambda})^{-1}\eta^{\alpha\widehat{\alpha}}(\lambda\gamma_a)_\alpha(\widehat{\lambda}\gamma_b)_{\widehat{\alpha}}\partial x^a\overline{\partial}x^b$. So classical instanton solutions satisfy

$$(\lambda\gamma_a)_\alpha\partial x^a = 0, \quad (\widehat{\lambda}\gamma_a)_{\widehat{\alpha}}\overline{\partial}x^a = 0, \tag{5.1}$$

where $(\lambda\gamma_a)_\alpha$ determines which five complex components of ∂x^a must vanish.

Another new feature of the topological sigma model of (4.1) is that the ghost-number anomaly does not fix the number of unintegrated versus integrated vertex operators. Since vertex operators can be multiplied by inverse powers of $(\eta\lambda\widehat{\lambda})$ without spoiling BRST invariance, one can construct unintegrated vertex operators of ghost-number zero such as $V = (\eta\lambda\widehat{\lambda})^{-1}\lambda^\alpha\widehat{\lambda}^{\widehat{\alpha}}A_{\alpha\widehat{\alpha}}(x,\theta,\widehat{\theta})$. It is unclear if the topological amplitude prescription should involve both unintegrated and integrated vertex operators, or only unintegrated vertex operators. Similarly, it is unclear if the genus g topological amplitude prescription requires integration over the moduli of genus g Riemann surfaces.

In addition to describing the zero radius $AdS_5 \times S^5$ limit, the topological model of (4.1) can also be interpreted as a tensionless string in which all massless and massive background fields are treated on equal footing. Changing the target-space metric in the topological action is a BRST-trivial operation so, as proposed by Witten, the topological model describes string theory in an "unbroken phase" in which general covariance does not require an explicit metric [19, 21].

By giving background values to physical moduli, one can perturb the topological model into non-topological string theories which describe backgrounds that are asymptotically $AdS_5 \times S^5$ but are not necessarily PSU(2,2|4) invariant. For example, perturbing with the vertex operator for the radius modulus deforms the topological action into the PSU(2,2|4)-invariant $AdS_5 \times S^5$ action of (2.11), but perturbing with other physical moduli will lead to superstring backgrounds which are asymptotically $AdS_5 \times S^5$ but which are not PSU(2,2|4) invariant.

In some sense, these asymptotically $AdS_5 \times S^5$ backgrounds are more natural backgrounds for the pure spinor formalism than asymptotically flat backgrounds. In asymptotically $AdS_5 \times S^5$ backgrounds, the worldsheet action can always be constructed from the Metsaev-Tseytlin coset $g \in \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times \mathrm{SO}(5)}$ even though the action is not necessarily invariant under the global PSU(2,2|4) isometries $\delta g = \Sigma g$. Furthermore, the BRST operator in these backgrounds always acts geometrically as $Qg = g(\lambda^\alpha T_\alpha + \widehat{\lambda}^{\widehat{\alpha}} T_{\widehat{\alpha}})$ and there is no need to introduce non-minimal variables. And in the limit where the radius goes to zero, the topological $AdS_5 \times S^5$ pure spinor action and BRST operator can be derived by gauge-fixing a G/G principal chiral model. This contrasts with the pure spinor formalism in a flat background which has not yet been derived in a simple manner from gauge fixing.

Acknowledgments

I would like to thank Y. Aisaka, R. Gopakumar, P. Howe, J. Maldacena, A. Mikhailov, N. Nekrasov, A. Polyakov, W. Siegel, C. Vafa, B.C. Vallilo, H. Verlinde and E. Witten for useful discussions, and CNPq grant 300256/94-9 and FAPESP grant 04/11426-0 for partial financial support.

References

[1] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, *JHEP* **04** (2000) 018 [hep-th/0001035] [SPIRES].

[2] N. Berkovits, *Quantum consistency of the superstring in $AdS_5 \times S^5$ background*, *JHEP* **03** (2005) 041 [hep-th/0411170] [SPIRES].

[3] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028] [SPIRES].

[4] N. Berkovits, *Pure spinor formalism as an $N = 2$ topological string*, *JHEP* **10** (2005) 089 [hep-th/0509120] [SPIRES].

[5] N. Berkovits and N. Nekrasov, *Multiloop superstring amplitudes from non-minimal pure spinor formalism*, *JHEP* **12** (2006) 029 [hep-th/0609012] [SPIRES].

[6] N. Berkovits, *Conformal field theory for the superstring in a Ramond-Ramond plane wave background*, *JHEP* **04** (2002) 037 [hep-th/0203248] [SPIRES].

[7] R. Gopakumar, *From free fields to AdS*, *Phys. Rev.* **D 70** (2004) 025009 [hep-th/0308184] [SPIRES].

[8] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [hep-th/0312171] [SPIRES].

[9] N. Berkovits, *A new limit of the $AdS_5 \times S^5$ σ-model*, *JHEP* **08** (2007) 011 [hep-th/0703282] [SPIRES].

[10] N. Berkovits and C. Vafa, *Towards a worldsheet derivation of the Maldacena conjecture*, *JHEP* **03** (2008) 031 [*AIP Conf. Proc.* **1031** (2008) 21] [arXiv:0711.1799] [SPIRES].

[11] N. Berkovits, *Perturbative super-Yang-Mills from the topological $AdS_5 \times S^5$ σ-model*, *JHEP* **09** (2008) 088 [arXiv:0806.1960] [SPIRES].

[12] A. Polyakov, *Old and new aspects of the strings/gauge correspondence*, in proceedings of *Strings*, Cambridge U.K. July 15–20 2002, http://www.damtp.cam.ac.uk/strings02/avt/polyakov/.

[13] H. Verlinde, private communication.

[14] Y. Aisaka and Y. Kazama, *A new first class algebra, homological perturbation and extension of pure spinor formalism for superstring*, *JHEP* **02** (2003) 017 [hep-th/0212316] [SPIRES].

[15] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *JHEP* **04** (2002) 013 [hep-th/0202021] [SPIRES].

[16] D. Gaiotto and L. Rastelli, *A paradigm of open/closed duality: Liouville D-branes and the Kontsevich model*, *JHEP* **07** (2005) 053 [hep-th/0312196] [SPIRES].

[17] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [hep-th/9811131] [SPIRES].

[18] H. Ooguri and C. Vafa, *Worldsheet derivation of a large-N duality*, *Nucl. Phys.* **B 641** (2002) 3 [hep-th/0205297] [SPIRES].

[19] E. Witten, *Topological Quantum Field Theory*, *Commun. Math. Phys.* **117** (1988) 353 [SPIRES].

[20] E. Witten, *Topological σ-models*, *Commun. Math. Phys.* **118** (1988) 411 [SPIRES].

[21] E. Witten, *Spacetime and topological orbifolds*, *Phys. Rev. Lett.* **61** (1988) 670 [SPIRES].

[22] N. Nekrasov, private communication.

[23] N. Berkovits and P.S. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, *Nucl. Phys.* **B 635** (2002) 75 [hep-th/0112160] [SPIRES].

[24] I. Oda and M. Tonin, *On the Berkovits covariant quantization of GS superstring*, *Phys. Lett.* **B 520** (2001) 398 [hep-th/0109051] [SPIRES].

[25] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, *Superstring theory on $AdS_2 \times S^2$ as a coset supermanifold*, *Nucl. Phys.* **B 567** (2000) 61 [hep-th/9907200] [SPIRES].

[26] G. Boussard, private communication.

JHEP09(2009)051

[27] N. Berkovits and O. Chandía, *Superstring vertex operators in an $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 596** (2001) 185 [hep-th/0009168] [SPIRES].

[28] N. Berkovits, *Relating the RNS and pure spinor formalisms for the superstring*, *JHEP* **08** (2001) 026 [hep-th/0104247] [SPIRES].

[29] E. Witten, *Two-dimensional models with* $(0,2)$ *supersymmetry: perturbative aspects*, hep-th/0504078 [SPIRES].

[30] N. Berkovits, *Covariant quantization of the superparticle using pure spinors*, *JHEP* **09** (2001) 016 [hep-th/0105050] [SPIRES].

[31] M. Movshev and A.S. Schwarz, *On maximally supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 681** (2004) 324 [hep-th/0311132] [SPIRES].

[32] L. Brink, O. Lindgren and B.E.W. Nilsson, $N = 4$ *Yang-Mills theory on the light cone*, *Nucl. Phys.* **B 212** (1983) 401 [SPIRES].

[33] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637 [hep-th/9207094] [SPIRES].

[34] D. Gaiotto, N. Itzhaki and L. Rastelli, *Closed strings as imaginary D-branes*, *Nucl. Phys.* **B 688** (2004) 70 [hep-th/0304192] [SPIRES].

JHEP09(2009)051